Chapter 5

Interacting Fields

So far, we have dealt only with free fields. Even though we have seen many remarkable features such as the spin of fermions, microscopic causality and the Lorentz invariance of quantized fields, and so on, all measurable effects occur through interactions of fields primarily as decays and scatterings which we will discuss in this chapter.

Before going into details, let's take a rough look at what we will be dealing with. As an example, take the creation of Higgs particle by annihilation of a fermion pair:

$$f + \bar{f} \to H\,, \tag{5.1}$$

where the fermion could be any lepton or quark, and the Higgs is a neutral spin-0 particle predicted by the standard model of elementary particles. We can roughly picture this interaction in terms of non-quantized fields as follows: at the beginning we have two overlapping plane waves, one for f and the other for $\bar{f}$. Now, suppose that the overlap of the waves acts as source of the Higgs field. Namely, at every point of the overlap, Higgs field is created and spherically propagates outward and they linearly add up (Huygens' principle). As the end result of the sum of all the spherical waves propagating from every point of overlap, we will have a macroscopic wave of Higgs field coming out of the region of the overlap.

Of course, no macroscopic Higgs wave should be generated unless the invariant mass of the incoming fermion pair happens to be the mass of the Higgs (i.e. 4-momentum conservation):

$$(p_f + p_{\bar{f}})^2 = M^2\,, \tag{5.2}$$

where p_f and $p_{\bar{f}}$ are 4-momenta of the fermion and antifermion, respectively, and M is the mass of the Higgs particle. Such a constraint, as we will see, is automatically built into the wave picture: in short, the microscopic spherical waves from the sources do not add up constructively to form a macroscopic wave unless the above condition is met. Another constraint is that the spins of the fermion pair should properly add up to form a spin-0 particle. This specifies how the four components of the fermion

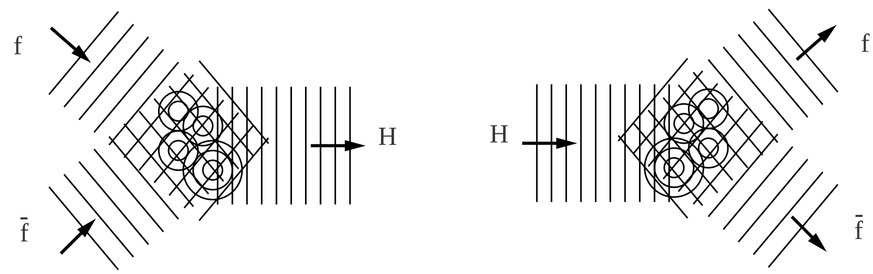


Figure 5.1: Rough graphical representations of the Higgs creation $f\,\bar{f} \to H$ and the Higgs decay $H \to f\,\bar{f}$.

fields should be combined to define the 'overlap', and largely determines the form of the interaction term in the Lagrangian. As we will see later, the existence of such a term in the Lagrangian in turn indicates that the Higgs field itself acts as a source of a fermion pair. It then represents a decay of the Higgs particle to a fermion pair:

$$H \to f\,\bar{f}\,, \tag{5.3}$$

which we will take in the following as an example to introduce the framework and techniques for evaluating interaction rates.

The above picture is not complete; we need to treat it in the framework of quantized field theory. What we are interested in is the probability amplitude for a given initial state be found in a certain final state, and it can in principle be obtained once we know the Hamiltonian of the system. In the Schrödinger picture, we would prepare the initial state and let it evolve according to the Schrödinger equation of motion and then take the inner product of the evolved state and the final state of interest. In the Heisenberg picture, we would find the state that represents the given initial-state at $t = -\infty$ and the state that represents the desired final-state at $t = +\infty$ and take their inner product. Here, we will adopt an intermediate picture, called the interaction picture, where the rapid oscillations of free fields that do not change the physical content (such as particle types and 4-momenta) are contained in the field operators and the states evolve relatively slowly reflecting the change in the physical content. Then the transition amplitude is given by taking the inner product of the evolved state and the given final state as in the Schrödinger picture. Let us now start from examining what form of interaction is possible for the Higgs-fermion coupling.

5.1 Lagrangian for the decay $H \to f\bar{f}$

The 'overlap' of two fermion fields ψ_1 and ψ_2 can be written in general as

$$\bar{\psi}_1 \Gamma \psi_2\,, \tag{5.4}$$

where Γ is an arbitrary 4×4 matrix. We have seen that such quantity can be expressed as a linear combination of *bilinear covariants* which transform under Lorentz transformation in well-defined ways:

$$\bar{\psi}_1\psi_2,\ \bar{\psi}_1\gamma^\mu\psi_2,\ \bar{\psi}_1\sigma^{\mu\nu}\psi_2,\ \bar{\psi}_1\gamma_5\gamma^\mu\psi_2,\ \bar{\psi}_1\gamma_5\psi_2\,. \tag{5.5}$$

As we have seen in (4.84), in order for the equation of motion to be Lorentz-invariant, the Lagrangian should be a Lorentz scalar. The simplest choices are

$$\phi\,\bar{\psi}_1\psi_2, \quad \text{or} \quad \phi\,\bar{\psi}_1\gamma_5\psi_2\,, \tag{5.6}$$

where ϕ is the spin-0 Higgs field. Note that the bilinear covariant has the same transformation property under proper and orthochronous transformation as the Higgs field; namely, it is a scalar or a pseudoscalar. This is because when the system is viewed in a different Lorentz frame, the relevant overlap should transform in the same way as the field created by the source in order for the same wave-source picture to be valid in the new frame. Which of the two forms to take depends on the transformation property of the Higgs field under parity and whether the process in question is invariant under parity. Here, we assume the proper overlap is a scalar. Since the fermion and its antiparticle are represented by the same field (as e^- and e^+ are representated by a single Dirac field ψ), the interaction term is then

$$\mathcal{L}_{\text{int}} = \lambda\,\phi\,\bar{\psi}\psi\,. \tag{5.7}$$

where λ is a real constant that specifies the strength of the source for a given overlap, and is called the coupling constant. Note that the interaction term is hermitian; if it were not hermitian, then its hermitian conjugate would have to be added to make the whole hermitian.

What is the dimension of the coupling constant λ? To find it, we need to know the dimensions of the fields ϕ and ψ. Since $c = 1$, mass (m) and energy-momentum (p^μ) have the same dimension:

$$\dim(m) = \dim(P^\mu) \overset{\text{def}}{\equiv} E \quad (\mu = 0, 1, 2, 3)\,. \tag{5.8}$$

Since $p \cdot x$ appears in exponents of exponentials ($e^{-ip\cdot x}$), $p \cdot x$ should be dimensionless; thus, x^μ should have dimension E^{-1}:

$$\dim(x^\mu) = E^{-1} \quad (\mu = 0, 1, 2, 3)\,. \tag{5.9}$$

The dimension of total Lagrangian $L = \int d^3x\mathcal{L}$ is E since $L = T - V$; then the Lagrangian density $\mathcal{L}$ should have dimension E^4:

$$\dim(d^3x\mathcal{L}) = E \quad \to \quad \dim(\mathcal{L}) = E^4\,. \tag{5.10}$$

The term $m^2\phi^2$ in the Klein-Gordon Lagrangian $\mathcal{L} = \frac{1}{2}(\partial_\mu\phi\partial^\mu\phi - m^2\phi^2)$ indicates that

$$\dim(m^2\phi^2) = E^4 \quad \rightarrow \quad \dim(\phi) = E \quad (\phi: \text{ scalar field}), \tag{5.11}$$

and the term $m\bar{\psi}\psi$ in $\mathcal{L} = \bar{\psi}(i\partial\!\!\!/ - m)\psi$ gives the dimension of fermion field:

$$\dim(m\bar{\psi}\psi) = E^4 \quad \rightarrow \quad \dim(\psi) = E^{\frac{3}{2}} \quad (\psi: \text{ fermion field}). \tag{5.12}$$

Then, in order for the interaction terms to have dimension E^4, λ has to be dimensionless:

$$\dim(\lambda\,\phi\,\bar{\psi}\psi) = E^4 \quad \rightarrow \quad \dim(\lambda) = E^0. \tag{5.13}$$

The Lagrangian density of the system is sum of free-field terms and the interaction term:

$$\boxed{\begin{gathered} \mathcal{L} = \mathcal{L}_H + \mathcal{L}_f + \mathcal{L}_{\text{int}}, \\ \mathcal{L}_H = \frac{1}{2}(\partial_\mu\phi\partial^\mu\phi - M^2\phi^2), \quad \mathcal{L}_f = \bar{\psi}(i\partial\!\!\!/ - m)\psi, \\ \mathcal{L}_{\text{int}} = \lambda\,\phi\,\bar{\psi}\psi. \end{gathered}} \tag{5.14}$$

The equation of motion for the scalar field can be obtained from the Lagrangian density using the Euler-Lagrange equation:

$$\underbrace{\frac{\partial\mathcal{L}}{\partial\phi}}_{-M^2\phi + \lambda\bar{\psi}\psi} = \underbrace{\partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}}_{\partial^2\phi} \quad \rightarrow \quad (\partial^2 + M^2)\phi = \lambda\,\bar{\psi}\psi, \tag{5.15}$$

which is probably a more familiar form that shows that the quantity $\bar{\psi}\psi$ is acting as a source of the scalar field.

The conjugate fields are the same as before:

$$\pi \equiv \frac{\partial\mathcal{L}}{\partial\dot{\phi}} = \dot{\phi}, \quad \pi_n^f \equiv \frac{\partial\mathcal{L}}{\partial\dot{\psi}_n} = i\psi_n^\dagger, \tag{5.16}$$

where the superscript f of π^f indicates that it is the field conjugate to the fermion field. The Hamiltonian density of the system is then

$$\begin{aligned} \mathcal{H} &\equiv \sum_k \pi_k\dot{\phi}_k - \mathcal{L} \\ &= (\pi\dot{\phi} - \mathcal{L}_H) + (\sum_n \pi_n^f\dot{\psi}_n - \mathcal{L}_f) - \mathcal{L}_{\text{int}} \\ &= \mathcal{H}_H + \mathcal{H}_f + \mathcal{H}_{\text{int}}, \end{aligned} \tag{5.17}$$

where $\mathcal{H}_H$ and $\mathcal{H}_f$ are the same free-field Hamiltonians as before, and

$$\boxed{\mathcal{H}_{\text{int}} = -\mathcal{L}_{\text{int}}}, \tag{5.18}$$

which is true in general as long as there is no time derivative of fields in $\mathcal{L}_{\rm int}$, in which case the definition of the conjugate fields would be altered and thus $\mathcal{H}$ would have extra terms.

Quantization proceeds as usual; namely, fields are now considered to be operators in the Heisenberg picture, and commutation and anticommutation relations are imposed as before among the fields. The fermion field and the scalar field are assumed to commute (not anticommute). Then, all time dependences are given by Heisenberg's equations of motion using the total Hamiltonian obtained above, and in principle we should be able to calculate the probability for a given initial configuration to end up as a given final configuration.

5.2 Interaction Picture and the Dyson Series

We have been using the *Heisenberg picture* in which all time dependences are in the operators and the states do not change with time. An alternative picture is the *Schrödinger picture* in which all time dependences are in the states and the operators do not vary with time. To deal with interactions, however, it is convenient to take an intermediate picture in which the operators (i.e., fields) vary according to the free-field Hamiltonian, and the states change according to the interaction part of the Hamiltonian - called the *interaction picture*. As we will see, these pictures are related by unitary transformations of states and operators:

$$|a'\rangle = V|a\rangle\,,\quad O' = VOV^\dagger\,. \tag{5.19}$$

where V is a time-dependent unitary operator. Since all measurements are given by matrix elements of some operators sandwiched between states, $\langle a|O|b\rangle$, the three approaches are effectively identical since matrix elements are invariant under unitary transformations:

$$\langle a'|O'|b'\rangle = \langle a|V^\dagger VOV^\dagger V|b\rangle = \langle a|O|b\rangle\,. \tag{5.20}$$

We start by reviewing the relations among the three pictures.

Interaction picture

The following discussion is valid for any closed quantum mechanical system; we keep in mind, however, that in our case the operators are fields ($\phi\,\bar\psi\psi$, ϕ^2, etc.) and the states belong to the Hilbert space ($a^\dagger_{\vec p}|0\rangle$, $b^\dagger_{\vec p,\vec s}|0\rangle$, etc.).

In the Heisenberg picture, the time dependences of any state $|a\rangle$ and any operator O_H are given by

$$\boxed{\dot{|a\rangle}_H = 0\,,\quad \dot O_H = i[H_H\,, O_H]}\quad \text{(Heisenberg picture)}\,, \tag{5.21}$$

where the subscript H indicates the Heisenberg picture.

Transition to the Schrödinger picture is achieved by the transformation

$$|a\rangle_S \stackrel{\text{def}}{\equiv} V|a\rangle_H\,, \quad O_S \stackrel{\text{def}}{\equiv} V O_H V^\dagger\,, \quad \boxed{V \stackrel{\text{def}}{\equiv} e^{-iH_H t}}\,, \tag{5.22}$$

where V is unitary since H_H is hermitian, and the subscript S indicates the Schrödinger picture. Since H_H commutes with $V = e^{-iH_H t} = \sum_n (-iH_H t)^n/n\,!$, we have

$$H_S \equiv \underbrace{V H_H}_{\text{commute}} V^\dagger = H_H \underbrace{V V^\dagger}_{1} = H_H\,, \quad \dot{H}_H = \dot{H}_S = 0\,, \tag{5.23}$$

which allows us to use $(\partial/\partial x)e^{xA} = Ae^{xA}$ (1.112) to obtain

$$\dot{V} = (-iH_H)V = V(-iH_H)\,. \tag{5.24}$$

Taking the time derivative of $|a\rangle_S = V|a\rangle_H$ and noting that $|\dot{a}\rangle_H = 0$, we get

$$|\dot{a}\rangle_S = \dot{V}|a\rangle_H = (-iH_H V)|a\rangle_H = -iH_H|a\rangle_S\,. \tag{5.25}$$

Similarly, taking the time derivative of $O_S = V O_H V^\dagger$,

$$\begin{aligned} \dot{O}_S &= \dot{V} O_H V^\dagger + V O_H \dot{V}^\dagger + V \dot{O}_H V^\dagger \\ &= \underbrace{V(-iH_H)O_H V^\dagger + V O_H (iH_H V^\dagger)}_{V\, i[O_H, H_H] V^\dagger} + V \underbrace{\dot{O}_H}_{i[H_H, O_H]} V^\dagger = 0\,. \end{aligned} \tag{5.26}$$

Thus, in the Schrödinger picture, the time dependences are given by

$$\boxed{|\dot{a}\rangle_S = -iH_H|a\rangle_S\,, \quad \dot{O}_S = 0} \quad \text{(Schrödinger picture)}\,, \tag{5.27}$$

namely, the operators are now constants and states change with time.

Now, we move to the interaction picture by starting from the Schrödinger picture and dividing the Hamiltonian into two parts

$$H_S = H_S^0 + h_S\,. \tag{5.28}$$

where the two pieces do not commute in general. Later, H_S^0 is taken to be the free field part and h_S the interaction part, but at present the division is arbitrary. Then, we apply a transformation by a unitary operator $V^0 = e^{iH_S^0 t}$:

$$|a\rangle_I \stackrel{\text{def}}{\equiv} V^0|a\rangle_S\,, \quad O_I \stackrel{\text{def}}{\equiv} V^0 O_S V^{0\dagger}\,, \quad \boxed{V^0 \stackrel{\text{def}}{\equiv} e^{iH_S^0 t}}\,, \tag{5.29}$$

where the subscript I indicates the interaction picture. Since $\dot{H}_S^0 = 0$, we can use (1.112) to get

$$\dot{V}^0 = iH_S^0 V^0 = V^0 iH_S^0\,. \tag{5.30}$$

Then, the time derivative of $|a\rangle_I = V^0|a\rangle_S$ becomes

$$\begin{aligned}
|\dot{a}\rangle_I &= \underbrace{\dot{V}^0}_{V^0 iH_S^0}|a\rangle_S + V^0 \underbrace{|\dot{a}\rangle_S}_{-iH_S|a\rangle_S \text{ by (5.27)}} \\
&= iV^0 \overbrace{(H_S^0 - \underbrace{H_S)\,|a\rangle_S}_{V^{0\dagger}V^0}}^{-h_S} = -i\underbrace{V^0 h_S V^{0\dagger}}_{h_I}\underbrace{V^0|a\rangle_S}_{|a\rangle_I}
\end{aligned} \tag{5.31}$$

and the time derivative of $O_I = V^0 O_S V^{0\dagger}$ is (using $\dot{O}_S = 0$)

$$\begin{aligned}
\dot{O}_I &= \dot{V}^0 O_S V^{0\dagger} + V^0 O_S \dot{V}^{0\dagger} \\
&= (iH_S^0 \underbrace{V^0)O_S V^{0\dagger}}_{O_I} + \underbrace{V^0 O_S\Big(V^{0\dagger}}_{O_I}(-iH_S^0)\Big) \\
&= i[H_S^0\,, O_I] = i[H_I^0\,, O_I]
\end{aligned} \tag{5.32}$$

In the last step, we have used

$$H_I^0 \equiv \underbrace{V^0 H_S^0}_{\text{commute}}\, V^{0\dagger} = H_S^0\,. \tag{5.33}$$

Thus, the time dependences in the interaction picture are

$$\boxed{|\dot{a}\rangle_I = -ih_I|a\rangle_I\,, \quad \dot{O}_I = i[H_I^0\,, O_I]} \quad \text{(interaction picture)}\,, \tag{5.34}$$

At this point, we take the free-field part of the Hamiltonian as H^0, and the interaction part as h. If there is no explicit time derivatives in these terms, they have the same form in all three pictures. For example,

$$\begin{aligned}
h_H &= \int d^3x\, \phi_H \bar{\psi}_H \psi_H\,, \\
h_S &= V\int d^3x\, \phi_H (V^\dagger V)\bar{\psi}_H (V^\dagger V)\psi_H V^\dagger = \int d^3x\, \phi_S \bar{\psi}_S \psi_S\,, \\
h_I &= V^0\int d^3x\, \phi_S (V^{0\dagger} V^0)\bar{\psi}_S (V^{0\dagger} V^0)\psi_S V^{0\dagger} = \int d^3x\, \phi_I \bar{\psi}_I \psi_I\,.
\end{aligned} \tag{5.35}$$

If there are time derivatives in h, the time dependences of the unitary operators V and V^0 can in principle generate extra terms and break the above form invariance. The time derivatives of fields, however, are usually replaced by the conjugate fields in the Hamiltonian (for example, $\dot{\phi}$ is replaced by π) and thus h does not contain time derivatives of fields even when $\mathcal{L}_{\rm int}$ does.

In the interaction picture, fields have the same time dependence as that of free fields, and thus they can be expanded using the same normal-mode functions as before with creation and annihilation operators that do not depend on time. To see this more clearly, let's work it out for the $H - f\bar{f}$ Lagrangian (5.14). We have imposed the commutation relations in the Heisenberg picture; we note, however, that the commutation relations are invariant under change of pictures:

$$\begin{aligned} [A_H, B_H] = C_H \;\to\; V \times &\Big(A_H \underbrace{}_{V^\dagger V} B_H - B_H \underbrace{}_{V^\dagger V} A_H = C_H \Big) \times V^\dagger \\ \to\; [A_S, B_S] = C_S\,, &\quad \xrightarrow{\text{similarly}} [A_I, B_I] = C_I\,. \end{aligned} \tag{5.36}$$

Similarly, anticommutation relations are also invariant under change of pictures. Thus, for example, the commutation relations for the scalar field in the interaction picture are

$$[\phi_I(t,\vec{x})\,, \pi_I(t,\vec{x}\,')] = i\delta^3(\vec{x}-\vec{x}\,')\,, \quad \text{all others} = 0\,. \tag{5.37}$$

The free-field Hamiltonian in the interaction picture has the same form as in the Heisenberg picture as seen in (5.35):

$$H_I^0 \equiv \int d^3x \mathcal{H}_I^0\,, \quad \mathcal{H}_I^0 = \frac{1}{2}(\pi_I^2 + (\vec{\nabla}\phi_I)^2 + m^2\phi_I^2)\,. \tag{5.38}$$

Then, following exactly the same steps as in the case of free fields, the commutation relations and the equation of motion $\dot{O}_I = i[H_I^0\,, O_I]$ leads to

$$\pi_I = \dot{\phi}_I\,, \quad (\partial^2 + m^2)\phi_I = 0\,. \tag{5.39}$$

Thus, it satisfies the free field Klein-Gordon equation, and again following exactly the same procedure as before, the field ϕ_I can be momentum-expanded as

$$\phi_I(x) = \sum_{\vec{p}} (a_{\vec{p}I} e_{\vec{p}}(x) + a_{\vec{p}I}^\dagger e_{\vec{p}}^*(x))\,, \tag{5.40}$$

where $a_{\vec{p}}$'s are constants of motion, $e_{\vec{p}}(x)$ is the same normal-mode function as before, and the commutation relations among fields lead to those among $a_{\vec{p}}$'s:

$$[a_{\vec{p}I}\,, a_{\vec{p}\,'I}^\dagger] = \delta_{\vec{p},\vec{p}\,'}\,, \quad [a_{\vec{p}I}\,, a_{\vec{p}\,'I}] = [a_{\vec{p}I}^\dagger\,, a_{\vec{p}\,'I}^\dagger] = 0\,. \tag{5.41}$$

We then define the lowest energy state as the vacuum $|0\rangle$ and identify $a_{\vec{p}}^\dagger|0\rangle$ as the state where there is one scalar particle with momentum $\vec{p}$ in the entire universe. The situation is the same for fermion fields. Note that the corresponding commutation relations among a's in other pictures also hold since commutators are 'invariant' as we have seen.

The advantage of the interaction picture is that the operator fields carry the rapid phase oscillations of the type $e^{-ip\cdot x}$ which do not change the physical quantities such as particle type, energies and momenta, spins, etc., and the states evolve much slower representing changes in physical contents. For the rest of the chapter, we will stay in the interaction picture and drop the subscript I. The differential equation for states

$$|\dot{a}\rangle = -ih(t)|a\rangle\,, \tag{5.42}$$

can then be solved for a small h. Suppose we have an initial state $|i\rangle$ at $t = 0$. Then, the change in a small time dt is $-ih(t)dt\,|i\rangle$, and as long as the sum total of change is much smaller than $|i\rangle$, it will linearly add up over some finite time T (even though h itself in general oscillates rapidly covering many periods in the duration T):

$$|a(T)\rangle \approx |i\rangle - i\Big(\int_0^T h(t)dt\Big)|i\rangle\,, \tag{5.43}$$

Assuming that initial and final states are orthogonal and normalized as

$$\langle i|i\rangle = 1\,, \quad \langle f|f\rangle = 1\,, \tag{5.44}$$

the amplitude to find a given state $|f\rangle$ at time T is given by

$$\langle f|a(T)\rangle = -i\langle f|\int_0^T h(t)dt\,|i\rangle = -i\int_0^T dt\int d^3x\langle f|\mathcal{H}_{\rm int}(x)|i\rangle\,, \tag{5.45}$$

where we have used $h(t) \equiv \int d^3x\mathcal{H}_{\rm int}(x)$. This is the first order transition matrix element. In order to find the answer to all orders, we have to proceed more systematically.

U and S operators

Each of the basis states of the Hilbert space at time t_0, $a^\dagger_{\vec{p}}|0\rangle$, $a^\dagger_{\vec{p},\vec{s}}b^\dagger_{\vec{p}',\vec{s}'}|0\rangle$ etc., will evolve according to $|\dot{a}\rangle = -ih|a\rangle$, and at a later time t they will become some linear combinations of the original basis states with some complex coefficients. These sets of coefficients form a gigantic matrix and define an operator which transforms any state at $t = 0$ to a state at a later time t:

$$|a(t)\rangle = U(t,t_0)|a(t_0)\rangle\,, \tag{5.46}$$

which is sometimes called the evolution operator, or the U operator (or the U matrix). Clearly, $U(t_0,t_0)$ is the identity

$$U(t_0,t_0) = I\,. \tag{5.47}$$

Substituting $|a(t)\rangle = U(t,t_0)|a(t_0)\rangle$ in $|\dot{a}\rangle = -ih|a\rangle$,

$$\dot{U}(t,t_0)|a(t_0)\rangle = -ihU(t,t_0)|a(t_0)\rangle\,. \tag{5.48}$$

Since this holds for any state $|a(t_0)\rangle$, we have an operator equation for $U(t, t_0)$:

$$\boxed{\dot{U}(t, t_0) = -ihU(t, t_0)}\,. \tag{5.49}$$

At t_0, $U(t_0, t_0) = I$ is obviously unitary: $U^\dagger(t_0, t_0)U(t_0, t_0) = 1$. Taking the hermitian conjugate of above, and using $h^\dagger = h$

$$\dot{U}^\dagger = iU^\dagger h^\dagger = iU^\dagger h\,. \tag{5.50}$$

Then, $U^\dagger U$ is seen to be constant of motion:

$$\frac{d}{dt}(U^\dagger U) = \underbrace{\dot{U}^\dagger}_{iU^\dagger h}\, U + U^\dagger \underbrace{\dot{U}}_{-ihU} = 0\,. \tag{5.51}$$

Thus, $U^\dagger U = 1$ at any time; namely, $U(t, t_0)$ is always unitary.

The S operator (or S matrix) is defined by taking the limit $t_0 \to -\infty$ and $t \to \infty$:

$$\boxed{S \stackrel{\text{def}}{\equiv} \lim_{\substack{t_0 \to -\infty \\ t \to \infty}} U(t, t_0)}\,; \tag{5.52}$$

namely, it evolves the 'initial states' to the 'final states'. By 'infinite time', we actually mean some time duration T long enough to cover many oscillations of fields. In fact, we will later find that transition probabilities calculated perturbatively are proportional to T, and if T is truly taken to be infinity, the transition probabilities will diverge or linear approximation will break down.

Now, since U is unitary, so is S:

$$S^\dagger S = I\,. \tag{5.53}$$

As an initial state $|i\rangle$ will evolve to $S|i\rangle$ at $t = \infty$, the transition amplitude $i \to f$ is given by $\langle f|S|i\rangle$ and the probability to find the final state in $|f\rangle$ is then $|\langle f|S|i\rangle|^2$. The unitarity of S then reads

$$\langle i| \times \overbrace{S^\dagger S = I}^{}_{\textstyle\sum_f |f\rangle\langle f|} \times |i\rangle \quad \to \quad \sum_f |\langle f|S|i\rangle|^2 = 1\,, \tag{5.54}$$

where the sum is over all possible final states that are assumed to form a complete orthonormal set. Thus, the unitarity of S means that probability is conserved.

The next step is to solve the differential equation $\dot{U}(t, t_0) = -ihU(t, t_0)$ for a given interaction h. Our goal is to express U, and thus S, as a perturbation series in powers of h. In doing so, we have to be careful about the operator nature of h. Recall the definition of differentiation and integration of a matrix function:

$$\frac{dA(t)}{dt} \equiv \frac{A(t+dt)-A(t)}{dt}\,, \tag{5.55}$$

and

$$\int_{t_0}^{t_1} dtA(t) \equiv \sum_{n=1}^{N} A(t_0+ndt)dt \quad (t_1 - t_0 \equiv Ndt)\,. \tag{5.56}$$

Strictly following these definitions, we see that

$$\begin{aligned} \frac{d}{dt}\int_{t_0}^{t} dt' A(t') &\equiv \frac{1}{dt}\Big(\int_{t_0}^{t+dt} dt' A(t') - \int_{t_0}^{t} dt' A(t')\Big) \\ [\text{by (5.56)}\,] &= \frac{1}{dt}\,A(t)dt = A(t)\,; \end{aligned}$$

$$\to \boxed{\frac{d}{dt}\int_{t_0}^{t} dt' A(t') = A(t)}\,. \tag{5.57}$$

Then, *if* in general

$$\frac{d}{dt}e^{A(t)} = \dot{A}(t)e^{A(t)} \qquad (?)\,, \tag{5.58}$$

the solution of $\dot{U} = -ihU$ would be given by

$$U(t,t_0) = e^{-i\int_{t_0}^{t} dt' h(t')}\,, \tag{5.59}$$

as can be readily verified using (5.57). The relation (5.58), however, does not hold unless $[A,\dot{A}] = 0$:

$$\begin{aligned} \Big(\frac{d}{dt}e^{A(t)}\Big)dt &= \underbrace{e^{A(t+dt)}}_{e^{(A(t)+\dot{A}(t)dt)}\ \text{by (5.55)}} - e^{A(t)} \\ (\text{if } [A,\dot{A}]=0\ ) &= e^{A(t)}e^{\dot{A}(t)dt} - e^{A(t)} \quad [\text{ used (1.113) }] \\ &= e^{A(t)}\Big(\underbrace{e^{\dot{A}(t)dt}-1}_{\dot{A}(t)dt}\Big) \\ &= \dot{A}(t)e^{A(t)}dt\,. \end{aligned} \tag{5.60}$$

Earlier, we have used $(\partial/\partial x)e^{xA} = Ae^{xA}$ for a constant operator A which is consistent with the above observation since

$$[xA, \frac{d}{dx}(xA)] = [xA, A] = 0\,. \tag{5.61}$$

In general, however, the condition is not satisfied as can be seen in $[\phi(t,\vec{x}),\dot{\phi}(t,\vec{x}')]\neq 0$. We will now show by direct substitution that the solution is given by

$$\begin{aligned} U(t,t_0) = 1 \ &+\ (-i)\int_{t_0}^{t} dt_1 h(t_1) + \ldots \\ &+\ (-i)^n \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 \cdots \int_{t_0}^{t_{n-1}} dt_n\, h(t_1)h(t_2)\cdots h(t_n)\ + \ldots \end{aligned} \tag{5.62}$$

Taking the time derivative of the above and using (5.57),

$$\dot{U}(t,t_0) = (-i)h(t) + \ldots + (-i)^n \int_{t_0}^{t} dt_2 \cdots \int_{t_0}^{t_{n-1}} dt_n\, h(t)h(t_2)\cdots h(t_n)\ + \ldots \tag{5.63}$$

Taking $h(t)$ in the integrand out in front and relabeling ($t_i \to t_{i-1}$),

$$\begin{aligned} \dot{U}(t,t_0) &= -ih(t)\Big(1 + \ldots + (-i)^{n-1}\int_{t_0}^{t} dt_1 \cdots \int_{t_0}^{t_{n-2}} dt_{n-1}\, h(t_1)\cdots h(t_{n-1}) + \ldots\Big) \\ &= -ih\, U(t,t_0)\,, \end{aligned} \tag{5.64}$$

which shows that the series (5.62) is indeed a solution of $\dot{U} = -ihU$.

It will be convenient later if we express the solution using the same integration range (t_0,t) for all integrals. To do so, we note

$$I_n \stackrel{\text{def}}{\equiv} \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 \cdots \int_{t_0}^{t_{n-1}} dt_n\, h(t_1)\cdots h(t_n) = \int_{V_n(t_1>\cdots>t_n)} dt_1\cdots dt_n\, h(t_1)\cdots h(t_n)\,, \tag{5.65}$$

where V_n is the n-dimensional cube defined by $t_0 < t_i < t$ ($i = 1,\ldots,n$), and the integration range $V_n(t_1 > \cdots > t_n)$ is its sub-volume limited to $(t_1 > \cdots > t_n)$. Relabeling $(t_1,\ldots,t_n) \to (t_{i_1},\ldots,t_{i_n})$, where $(i_1,\ldots,i_n)$ is any permutation of $(1,\ldots,n)$,

$$\begin{aligned} I_n &= \int_{V_n(t_{i_1}>\ldots>t_{i_n})} dt_1\cdots dt_n\, h(t_{i_1})\cdots h(t_{i_n}) \\ &= \int_{V_n(t_{i_1}>\ldots>t_{i_n})} dt_1\cdots dt_n\, T\Big(h(t_1)\cdots h(t_n)\Big)\,, \end{aligned} \tag{5.66}$$

where we have defined *time-ordered product* by reordering of operators in the descending order of times:

$$T\Big(A_1(t_1)\cdots A_n(t_n)\Big) \stackrel{\text{def}}{\equiv} A_{i_1}(t_{i_1})\cdots A_{i_n}(t_{i_n}) \quad (t_{i_1} \geq \cdots \geq t_{i_n})\,. \tag{5.67}$$

For completeness, operators with the same time are defined to keep the original order. This procedure is understood to be a simple re-ordering of operators that are functions of time as they are written, and one should not redefine the functions before the time-ordering is done. For example,

$$A(1.5)B(2.0) = A'(2.5)B(2.0) \quad \text{if } A(t) \stackrel{\text{def}}{\equiv} A'(t+1)\,, \tag{5.68}$$

but

$$\begin{aligned} T(A(1.5)B(2.0)) &= B(2.0)A(1.5) \quad \text{flips the ordering,} \\ T(A'(2.5)B(2.0)) &= A'(2.5)B(2.0) \quad \text{does not}\,, \end{aligned} \tag{5.69}$$

and the results are in general different.

In the case at hand, we have $A_i(t) = h(t)$ for all i. Noting that sum of the regions $V_n(t_{i_1} > \ldots > t_{i_n})$ with all possible permutations ($n!$ of them) is just V_n, and that for each permutation, I_n is expressed as (5.66) with the *same integrand* $T(A_1(t_1)\cdots A_n(t_n))$, we have

$$\begin{aligned} n!I_n &= \sum_{\substack{\text{all}\\ \text{perm.}}} \int_{V_n(t_{i_1}>\ldots>t_{i_n})} dt_1\cdots dt_n\, T\Big(h(t_1)\cdots h(t_n)\Big) \\ &= \int_{V_n} dt_1\cdots dt_n\, T\Big(h(t_1)\cdots h(t_n)\Big)\,, \\ \to\ I_n &= \frac{1}{n!}\int_{t_0}^{t} dt_1\cdots\int_{t_0}^{t} dt_n\, T\Big(h(t_1)\cdots h(t_n)\Big)\,. \end{aligned} \tag{5.70}$$

Thus, the solution (5.62) is now written as

$$U(t,t_0) = 1 + \sum_{n=1}^{\infty}\frac{(-i)^n}{n!}\int_{t_0}^{t} dt_1\cdots\int_{t_0}^{t} dt_n\, T\Big(h(t_1)\cdots h(t_n)\Big)\,. \tag{5.71}$$

The S matrix is then obtained by taking the limit $t\to\infty$ and $t_0\to-\infty$.

$$\boxed{S = 1 + \sum_{n=1}^{\infty}\frac{(-i)^n}{n!}\int_{-\infty}^{\infty} dt_1\cdots\int_{-\infty}^{\infty} dt_n\, T\Big(h(t_1)\cdots h(t_n)\Big)}\,, \tag{5.72}$$

which is known as the Dyson series. The advantage of such an expansion is that often the interaction term is proportional to a small coupling constant and the higher-order terms become progressively insignificant.

Incidentally, the time-ordered product can be defined for cases where fermion fields are included:

$$\boxed{T(A_1(t_1)\cdots A_n(t_n)) \stackrel{\text{def}}{\equiv} s_P A_{i_1}(t_{i_1})\cdots A_{i_n}(t_{i_n}) \quad (t_{i_1}\ge\cdots\ge t_{i_n})}\,, \tag{5.73}$$

where $A_i(t)$ are time-dependent operators and s_P is $+1(-1)$ if the number of swaps of fermion operators needed for the re-ordering is even (odd). With this definition, the difference between before and after the time-ordering becomes a c-number as in the case of normal ordering. In deriving the Dyson series, we needed $s_P = +1$ in (5.66) which is actually the case even if fermion fields are involved since fermion fields always appear in pairs in the interaction term h.

5.3 Evaluation of the decay rate $H \to f\bar{f}$

Taking only the first-order term in the Dyson series,

$$S = -i \int_{-\infty}^{\infty} dt\, h(t) = -i \int_{-\infty}^{\infty} dt \int d^3x\, \mathcal{H}_{\text{int}}\,, \tag{5.74}$$

which is the same as the earlier result (5.45) if T is taken to be large ('infinity'). Assuming that there is no time derivatives in the interaction term, we have $\mathcal{H}_{\text{int}} = -\mathcal{L}_{\text{int}}$ (5.18), and

$$\boxed{S = i \int d^4x\, \mathcal{L}_{\text{int}}} \quad \text{(first order)}\,. \tag{5.75}$$

In our case, $\mathcal{L}_{\text{int}} = \lambda\phi\,\bar{\psi}\psi$ (5.14), where normal ordering is implicit, and thus,

$$S_{fi} = \langle f|S|i\rangle = i\lambda \int d^4x\, \langle f|\phi\,\bar{\psi}\psi|i\rangle\,. \tag{5.76}$$

Let's define the spins and 4-momenta of the particles involved as

$$\begin{array}{rl} H: & \quad P = (P^0, \vec{P})\,,\ \ P^0 = \sqrt{\vec{P}^{\,2} + M^2}\,, \\ f: & \vec{s}_1\,,\ p_1 = (p_1^0\,, \vec{p}_1)\,,\ \ p_1^0 = \sqrt{\vec{p}_1^{\,2} + m^2}\,, \\ \bar{f}: & \vec{s}_2\,,\ p_2 = (p_2^0\,, \vec{p}_2)\,,\ \ p_2^0 = \sqrt{\vec{p}_2^{\,2} + m^2}\,. \end{array} \tag{5.77}$$

Then, the initial and final states are

$$\begin{aligned} |i\rangle &= a^\dagger_{\vec{P}}|0\rangle\,, \\ |f\rangle &= a^\dagger_{\vec{p}_1,\vec{s}_1} b^\dagger_{\vec{p}_2,\vec{s}_2}|0\rangle\,, \end{aligned} \tag{5.78}$$

where if annihilation or creation operators have only a momentum subscript, they are understood to be for H, and if they have momentum and spin subscripts, then they are for f or $\bar{f}$.

We will now use momentum expansions of fields to write the matrix element S_{fi} in terms of creation and annihilation operators. Since S_{fi} is just a number, all creation and annihilation operators should disappear in the end and we will be left with numbers such as $e_{\vec{P}}$, $\bar{f}_{\vec{p},\vec{s}}$, and $g_{\vec{p},\vec{s}}$. With momentum expansions, we have

$$\begin{aligned} \langle f|\phi\,\bar{\psi}\psi|i\rangle &= \overbrace{\langle 0|b_{\vec{p}_2,\vec{s}_2} a_{\vec{p}_1,\vec{s}_1}}^{\langle f|} \\ &\times \overbrace{\sum_{\vec{q}}(a_{\vec{q}} e_{\vec{q}} + a^\dagger_{\vec{q}} e^*_{\vec{q}})}^{\phi} \overbrace{\sum_{\vec{p},\vec{s}}(a^\dagger_{\vec{p},\vec{s}} \bar{f}_{\vec{p},\vec{s}} + b_{\vec{p},\vec{s}} \bar{g}_{\vec{p},\vec{s}})}^{\bar{\psi}} \overbrace{\sum_{\vec{p}',\vec{s}'}(a_{\vec{p}',\vec{s}'} f_{\vec{p}',\vec{s}'} + b^\dagger_{\vec{p}',\vec{s}'} g_{\vec{p}',\vec{s}'})}^{\psi} \overbrace{a^\dagger_{\vec{P}}|0\rangle}^{|i\rangle}\,. \end{aligned} \tag{5.79}$$

There are many terms in $\phi\,\bar{\psi}\psi$, but there is only one term that survives - the term with matching annihilation and creation operators that counter those in $|i\rangle$ and $\langle f|$. First, all terms that contain $a^\dagger_{\vec{q}}$ vanish since they commute with fermion operators on their left and face the vacuum $\langle 0|$. Also, all terms with $a_{\vec{q}}$ ($\vec{q} \neq \vec{P}$) vanish since they commute with all operators on their right to face $|0\rangle$, leaving $a_{\vec{P}} e_{\vec{P}}$ as the only term in ϕ that survives. Similarly, the only term that survives in ψ is $b^\dagger_{\vec{p}_2,\vec{s}_2} g_{\vec{p}_2,\vec{s}_2}$ and the only term that survives in $\bar{\psi}$ is $a^\dagger_{\vec{p}_1,\vec{s}_1} \bar{f}_{\vec{p}_1,\vec{s}_1}$. One subtlety may be the vanishing of terms with $b_{\vec{p},\vec{s}}$ in $\bar{\psi}$ which need to go beyond $b^\dagger_{\vec{p}',\vec{s}'}$ of ψ to face $|0\rangle$. Actually, $b_{\vec{p},\vec{s}}$'s are already to the right of $b^\dagger_{\vec{p}',\vec{s}'}$'s because of the implicit normal ordering. Even if normal ordering is not assumed, when $b_{\vec{p},\vec{s}}$ is moved past $b^\dagger_{\vec{p}',\vec{s}'}$ it leaves behind $\delta_{\vec{p},\vec{p}'}\delta_{\vec{s},\vec{s}'}$ and such term vanishes anyway since the annihilation operator $a_{\vec{p}_1,\vec{s}_1}$ in $\langle f|$ will face $|0\rangle$ to the right. Thus, we have

$$\begin{aligned}
\langle f|\phi\,\bar{\psi}\psi|i\rangle &= \langle 0|b_{\vec{p}_2,\vec{s}_2} a_{\vec{p}_1,\vec{s}_1} (a_{\vec{P}} e_{\vec{P}})(a^\dagger_{\vec{p}_1,\vec{s}_1} \bar{f}_{\vec{p}_1,\vec{s}_1})(b^\dagger_{\vec{p}_2,\vec{s}_2} g_{\vec{p}_2,\vec{s}_2}) a^\dagger_{\vec{P}}|0\rangle \\
&= \underbrace{\langle 0|b_{\vec{p}_2,\vec{s}_2} a_{\vec{p}_1,\vec{s}_1} a_{\vec{P}}\, a^\dagger_{\vec{p}_1,\vec{s}_1} b^\dagger_{\vec{p}_2,\vec{s}_2} a^\dagger_{\vec{P}}|0\rangle}_{1:\ \text{norm of } a^\dagger_{\vec{p}_1,\vec{s}_1} b^\dagger_{\vec{p}_2,\vec{s}_2} a^\dagger_{\vec{P}}|0\rangle} e_{\vec{P}} (\bar{f}_{\vec{p}_1,\vec{s}_1} g_{\vec{p}_2,\vec{s}_2}) \\
&= e_{\vec{P}} (\bar{f}_{\vec{p}_1,\vec{s}_1} g_{\vec{p}_2,\vec{s}_2})\,,
\end{aligned} \tag{5.80}$$

which is now explicitly just a number (a function of x). The transition amplitude S_{fi} is then

$$\begin{aligned}
S_{fi} &= i\lambda \int d^4x\, e_{\vec{P}}(x)\, \bar{f}_{\vec{p}_1,\vec{s}_1}(x) g_{\vec{p}_2,\vec{s}_2}(x) \\
&= i\lambda \int d^4x \frac{e^{-iP\cdot x}}{\sqrt{2P^0V}} \frac{\bar{u}_{\vec{p}_1,\vec{s}_1} e^{ip_1\cdot x}}{\sqrt{2p_1^0V}} \frac{v_{\vec{p}_2,\vec{s}_2} e^{ip_2\cdot x}}{\sqrt{2p_2^0V}} \\
&= \frac{i\lambda\, \bar{u}_{\vec{p}_1,\vec{s}_1} v_{\vec{p}_2,\vec{s}_2}}{\sqrt{(2P^0V)(2p_1^0V)(2p_2^0V)}} \underbrace{\int d^4x\, e^{i(p_1+p_2-P)\cdot x}}_{(2\pi)^4\delta^4(p_1+p_2-P)}\ .
\end{aligned} \tag{5.81}$$

The four-dimensional delta function arose from the phase terms of the normal-mode functions, and indicates that the transition amplitude is zero unless the energy-momentum conservation $P = p_1 + p_2$ is satisfied.

Let's define the Lorentz-invariant matrix element $\mathcal{M}$ by

$$S_{fi} \equiv \frac{(2\pi)^4\delta^4\Big(p_1+p_2-P\Big)}{\sqrt{(2P^0V)(2p_1^0V)(2p_2^0V)}}\,\mathcal{M}\,, \tag{5.82}$$

where $\mathcal{M}$ is

$$\mathcal{M} = i\lambda\, \bar{u}_{\vec{p}_1,\vec{s}_1} v_{\vec{p}_2,\vec{s}_2}\,, \tag{5.83}$$

which is a scalar bilinear covariant and thus Lorentz-invariant. The probability to find the final state f is then the square of the amplitude:

$$|S_{fi}|^2 = \frac{\Big[(2\pi)^4\delta^4(p_1+p_2-P)\Big]^2}{(2P^0V)(2p_1^0V)(2p_2^0V)}|\mathcal{M}|^2 \,. \tag{5.84}$$

What is this square of the delta function? This oddity came about because we pretended that we are integrating over infinite space and time even though we are actually dealing with finite space V and finite time T. Thus, let's recover the origin of the delta functions (for one of the two δ's) and write

$$\begin{aligned} \Big[(2\pi)^4\delta^4(p_1+p_2-P)\Big]^2 &= (2\pi)^4\delta^4(p_1+p_2-P)\underbrace{\int_T dt\int_V d^3x\, e^{i(p_1+p_2-P)\cdot x}}_{TV\,:\,(P=p_1+p_2)} \\ &= TV(2\pi)^4\delta^4(p_1+p_2-P)\,, \end{aligned} \tag{5.85}$$

where the delta function allowed us to set $P=p_1+p_2$ in the integrand. Thus, the transition probability is

$$|S_{fi}|^2 = \frac{(2\pi)^4\delta^4(p_1+p_2-P)TV}{(2P^0V)(2p_1^0V)(2p_2^0V)}|\mathcal{M}|^2 \,. \tag{5.86}$$

This is the probability to find the final state with particular discrete momenta $\vec{p}_1$ and $\vec{p}_2$.

What we usually need is the probability to find the final-state momenta in some ranges d^3p_1 and d^3p_2 which are small but still contain many discrete values of $\vec{p}_1$ and $\vec{p}_2$, which should be equal to the corresponding decay rate $d\Gamma$ times T. From $\sum_{\vec{p}} = V/(2\pi)^3 \int d^3p$, we have

$$\sum_{\vec{p}_i\in d^3p_i} = \frac{V}{(2\pi)^3}d^3p_i \qquad (i=1,2)\,. \tag{5.87}$$

The transition probability $d\Gamma\, T$ is then obtained by summing over all final states with $\vec{p}_1\in d^3p_1$ and $\vec{p}_2\in d^3p_2$:

$$\begin{aligned} d\Gamma\, T &= \sum_{\vec{p}_1\in d^3p_1}\ \sum_{\vec{p}_2\in d^3p_2}|S_{fi}|^2 \\ &= \frac{V}{(2\pi)^3}d^3p_1\,\frac{V}{(2\pi)^3}d^3p_2\,\frac{(2\pi)^4\delta^4(p_1+p_2-P)VT}{(2P^0V)(2p_1^0V)(2p_2^0V)}\,|\mathcal{M}|^2\,; \end{aligned} \tag{5.88}$$

namely, the decay rate is given by

$$d\Gamma = \frac{(2\pi)^4}{2P^0}\delta^4(p_1+p_2-P)\frac{d^3p_1}{(2\pi)^3 2p_1^0}\,\frac{d^3p_2}{(2\pi)^3 2p_2^0}\,|\mathcal{M}|^2\,, \tag{5.89}$$

or

$$\boxed{d\Gamma = \frac{(2\pi)^4}{2P^0} d\Phi_2 |\mathcal{M}|^2}\,, \tag{5.90}$$

with the 2-body Lorentz-invariant phase space $d\Phi_2$ defined as

$$\boxed{d\Phi_2 \stackrel{\text{def}}{\equiv} \delta^4(p_1 + p_2 - P)\frac{d^3p_1}{(2\pi)^3 2p_1^0}\frac{d^3p_2}{(2\pi)^3 2p_2^0}}\,. \tag{5.91}$$

In deriving this formula, we did not require that the mass of the two final-state particles are the same; thus, it is valid for the case the masses are different, in which case we have $p_i^0 \equiv \sqrt{\vec{p}_i^{\,2} + m_i^2}$ $(i = 1, 2)$.

The total decay rate is obtained by integrating over all final states ('phase space'):

$$\Gamma \equiv \frac{(2\pi)^4}{2P^0} \int d\Phi_2 |\mathcal{M}|^2. \tag{5.92}$$

That $d\Phi_2$ is Lorentz-invariant can be seen by the identity (4.286). What do we mean by the Lorentz-invariance of differential expressions? It means that if a Lorentz-invariant function is integrated with the differential piece $d\Phi_2$, then the resulting quantity will have the same numerical value in any frame. Since $\mathcal{M}$ is Lorentz-invariant, when integrated over all phase-space, the result $\int d\Phi_n |\mathcal{M}|^2$ will be Lorentz-invariant. Thus, the above expression for Γ indicates that the decay rate is inversely proportional to the energy, which is just the time dilation effect due to boost.

We now perform the integration over the phase space in the rest frame of the parent particle. Since H is spinless, there is no preferred spacial direction, and thus the decay should be uniform over 4π steradians in the rest frame of H. Later, we will indeed see that $|\mathcal{M}|^2$ does not depend on direction of $\vec{p}_1$ or $\vec{p}_2$ in the C.M. frame. Using the identity (4.286), we can change d^3p_1 to d^4p_1 and eliminate the 4-dimentional delta function by integrating over p_1. Keeping the two final state masses to be different and denoting the integrand as X,

$$\begin{aligned}
\int d\Phi_2 X &= \int \delta^4(p_1 + p_2 - P)\underbrace{\frac{d^3p_1}{(2\pi)^3 2p_1^0}}_{\delta(p_1^2 - m_1^2)\theta(p_1^0)\frac{d^4p_1}{(2\pi)^3} \text{ by (4.286)}}\frac{d^3p_2}{(2\pi)^3 2p_2^0}\, X \\
&= \int \delta(p_1^2 - m_1^2)\theta(p_1^0)\frac{d^3p_2}{(2\pi)^6 2p_2^0}\, X\Big|_{p_1 = P - p_2} \\
&= \int \delta((P - p_2)^2 - m_1^2)\theta(P^0 - p_2^0)\frac{d^3p_2}{(2\pi)^6 2p_2^0}\, X \\
&= \int \delta((M^2 + m_2^2 - 2Mp_2^0) - m_1^2)\theta(M - p_2^0)\frac{d^3p_2}{(2\pi)^6 2p_2^0}\, X\,,
\end{aligned} \tag{5.93}$$

where in the last step we have used $P = (M, \vec{0})$. Since the integrand does not depend on the direction of $\vec{p}_2$, we can write

$$d^3p_2 = 4\pi\breve{p}_2^2\, d\breve{p}_2 = 4\pi\breve{p}_2\, p_2^0 dp_2^0 \qquad (\breve{p}_2 \equiv |\vec{p}_2|)\,, \tag{5.94}$$

where we have denoted $|\vec{p}|$ as $\breve{p}$ in order to simplify the notation during the computation, and used

$$p_2^{0\,2} = \breve{p}_2^2 + m_2^2 \quad \to \quad p_2^0 dp_2^0 = \breve{p}_2 d\breve{p}_2\,. \tag{5.95}$$

We can then complete the integral using the property of the delta function (4.109):

$$\begin{aligned}\int d\Phi_2 X \;&=\; \int \underbrace{\delta(M^2 + m_2^2 - 2Mp_2^0 - m_1^2)}_{\frac{1}{2M}\delta\left(p_2^0 - \frac{M^2+m_2^2-m_1^2}{2M}\right) \text{ by (4.109)}}\, \theta(M - p_2^0)\, \underbrace{\frac{4\pi\breve{p}_2\, p_2^0 dp_2^0}{(2\pi)^6 2p_2^0}}_{\frac{\breve{p}_2 dp_2^0}{(2\pi)^5}}\, X \\ &=\; \frac{\breve{p}_2}{(2\pi)^5 2M} X \end{aligned} \tag{5.96}$$

where $\breve{p}_2 = \sqrt{p_2^{0\,2} - m_2^2}$ with $p_2^0 = (M^2 + m_2^2 - m_1^2)/2M$, and $\theta(M - p_2^0) = +1$ since p_2^0 as given is always smaller than M (the energy of the daughter particle is always smaller than that of the parent). Thus, the total decay rate in the C.M. system for a uniform decay is given by (with $X = (2\pi)^4|\mathcal{M}|^2/2P^0$)

$$\boxed{\Gamma = \frac{|\vec{p}|}{8\pi M^2}\, |\mathcal{M}|^2}\,, \tag{5.97}$$

where we have defined $|\vec{p}_1| = |\vec{p}_2| \equiv |\vec{p}|$.

We now move on to the evaluation of $|\mathcal{M}|^2$. Since we know the explicit forms of u, v spinors, in principle we can calculate $\mathcal{M} = i\lambda\, \bar{u}_{\vec{p}_1,\vec{s}_1} v_{\vec{p}_2,\vec{s}_2}$ for given momenta and spins. There is, however, much quicker and also representation-independent way. Suppose we are not measuring the spins of the final state. Then, decay rates to all possible spin states (there are 4 of them for given $\vec{p}_1$ and $\vec{p}_2$) should be incoherently summed:

$$\Gamma = \frac{|\vec{p}|}{8\pi M^2} \sum_{\vec{s}_1,\vec{s}_2} |\mathcal{M}|^2\,. \tag{5.98}$$

As we will see, when the matrix element squared is summed over the spins, we obtain a trace of certain 4×4 matrix in the spinor space which can be evaluated using representation-independent techniques:

$$\sum_{\vec{s}_1,\vec{s}_2} |\mathcal{M}|^2 \;=\; \lambda^2 \sum_{\vec{s}_1,\vec{s}_2} \underbrace{(\bar{u}_{\vec{p}_1,\vec{s}_1} v_{\vec{p}_2,\vec{s}_2})^*}_{\bar{v}_{\vec{p}_2,\vec{s}_2} u_{\vec{p}_1,\vec{s}_1}} (\bar{u}_{\vec{p}_1,\vec{s}_1} v_{\vec{p}_2,\vec{s}_2})$$

$$
\begin{aligned}
&= \lambda^2 \sum_{\vec{s}_2} \bar{v}_{\vec{p}_2,\vec{s}_2} \underbrace{(\sum_{\vec{s}_1} u_{\vec{p}_1,\vec{s}_1} \bar{u}_{\vec{p}_1,\vec{s}_1})}_{(\not{p}_1 + m)} v_{\vec{p}_2,\vec{s}_2} \\
\text{(write out components)} \quad &= \lambda^2 \sum_{\vec{s}_2} (\bar{v}_{\vec{p}_2,\vec{s}_2})_n (\not{p}_1 + m)_{nm} (v_{\vec{p}_2,\vec{s}_2})_m \\
&= \lambda^2 \underbrace{\sum_{\vec{s}_2} (v_{\vec{p}_2,\vec{s}_2})_m (\bar{v}_{\vec{p}_2,\vec{s}_2})_n}_{(\not{p}_2 - m)_{mn}} (\not{p}_1 + m)_{nm} \\
&= \lambda^2 \Big[(\not{p}_2 - m)(\not{p}_1 + m) \Big]_{mm} \\
&= \lambda^2 \mathrm{Tr} \Big[(\not{p}_2 - m)(\not{p}_1 + m) \Big] . \qquad (5.99)
\end{aligned}
$$

In general, the spin sum over the form $\bar{u}_{\vec{p},\vec{s}} M u_{\vec{p},\vec{s}}$ or $\bar{v}_{\vec{p},\vec{s}} M v_{\vec{p},\vec{s}}$, where M is any 4×4 matrix, results in a trace: following the last four lines above with $\not{p}_1 + m \to M$, we obtain

$$
\begin{aligned}
\sum_{\vec{s}} \bar{u}_{\vec{p},\vec{s}} M u_{\vec{p},\vec{s}} &= \mathrm{Tr}\left[(\not{p} + m) M \right] , \\
\sum_{\vec{s}} \bar{v}_{\vec{p},\vec{s}} M v_{\vec{p},\vec{s}} &= \mathrm{Tr}\left[(\not{p} - m) M \right] .
\end{aligned} \qquad (5.100)
$$

The expression (5.99) can be readily evaluated using the trace theorems (Exercise 5.1). Since traces of an odd number of gamma matrices are zero,

$$
\begin{aligned}
\sum_{\vec{s}_1,\vec{s}_2} |\mathcal{M}|^2 &= \lambda^2 \Big[\underbrace{\mathrm{Tr} \not{p}_2 \not{p}_1}_{4 p_2 \cdot p_1} - m^2 \underbrace{\mathrm{Tr} I}_{4} \Big] \\
&= 4 \lambda^2 (p_2 \cdot p_1 - m^2) . \qquad (5.101)
\end{aligned}
$$

In the Higgs C.M. frame, we have $p_1 \equiv (E, \vec{p})$ and $p_2 = (E, -\vec{p})$; thus,

$$
p_2 \cdot p_1 - m^2 = E^2 + \vec{p}^{\,2} - m^2 = 2 \vec{p}^{\,2} . \qquad (5.102)
$$

Then, using this in (5.98), we finally obtain

$$
\Gamma = \frac{\lambda^2 |\vec{p}\,|^3}{\pi M^2} . \qquad (5.103)
$$

We can now evaluate the actual decay rate of the Higgs decay to a top quark and its antiquark. The mass of top quark m_t is 175 GeV, and let's assume that the Higgs mass is 450 GeV. Then $|\vec{p}| \sim 141$ GeV$/c$. In the standard model, the coupling constant λ is given by

$$
\lambda = \frac{g m_t}{2 m_W} , \qquad (5.104)
$$

where $g \sim 0.65$ is a universal coupling constant of the standard model and $m_W \sim 80$ GeV is the W boson mass. The numerical value of λ is thus ~ 0.71. There is one more complication; namely, quarks come in three colors, and each one is created with the same decay rate (5.103) thus increasing the rate by factor three. Putting all together, the decay rate $H \to t\bar{t}$ for a 450 GeV Higgs would be

$$\Gamma(H \to t\bar{t}) = \frac{3\lambda^2 |\vec{p}|^3}{\pi M^2} = 6.7\text{GeV}\,. \tag{5.105}$$

Let's reflect upon the procedures we just followed. We started from the first-order expression of the S operator and sandwiched it between the initial and final states which led to a space-time integral of $\langle f|\mathcal{L}_{\rm int}|i\rangle$. When momentum expansions were used for the fields appearing in $\mathcal{L}_{\rm int}$, only the term that contained the creation and annihilation operators that exactly matched those of the initial and final states survived. In short, the interaction term had to annihilate the particles in the initial state and create the particles in the final state. More specifically, for the creation operator of the initial state $a^\dagger_{\vec{P}}$, the term had to contain the annihilation operator $a_{\vec{P}}$ and the associated normal-mode function $e_{\vec{P}}(x)$ survived into S_{fi}. Similarly, the fermion (antifermion) in the final state came with the annihilation operator $a_{\vec{p}_1,\vec{s}_1}$ ($b_{\vec{p}_2,\vec{s}_2}$) and the surviving term in the interaction had to contain the matching creation operator $a^\dagger_{\vec{p}_1,\vec{s}_1}$ ($b^\dagger_{\vec{p}_2,\vec{s}_2}$) which contributed the normal-mode function $\bar{f}_{\vec{p}_1,\vec{s}_1}$ ($g_{\vec{p}_2,\vec{s}_2}$) to S_{fi}.

These contributions of the initial and final state particles to S_{fi} are valid for a general transition from an arbitrary number of initial state particles to an arbitrary number of final state particles. We did assume that the creation operators $a^\dagger_{\vec{p},\vec{s}}$ of fermion and the annihilation operators $b_{\vec{p},\vec{s}}$ of antifermion appear in the form $\bar{\psi}$ (and not as $\psi^\dagger$ which can be written as $\bar{\psi}\gamma^0$ in any case) as is the case for any bilinear covariant. These can be readily extended to scalars in the final state and fermions and antifermions in the initial state. Thus, the contributions to S_{fi} from the initial and final states are:

$$\begin{array}{lll}
e_{\vec{p}}(x) \text{ for a scalar in } |i\rangle & (a_{\vec{p}}\, e_{\vec{p}}(x) \text{ to match } a^\dagger_{\vec{p}}|0\rangle\,.) \\
e^*_{\vec{p}}(x) \text{ for a scalar in } |f\rangle & (a^\dagger_{\vec{p}}\, e^*_{\vec{p}}(x) \text{ to match } \langle 0|a_{\vec{p}}\,.) \\
f_{\vec{p},\vec{s}}(x) \text{ for a fermion in } |i\rangle & (a_{\vec{p},\vec{s}}\, f_{\vec{p},\vec{s}}(x) \text{ to match } a^\dagger_{\vec{p},\vec{s}}|0\rangle\,.) \\
\bar{f}_{\vec{p},\vec{s}}(x) \text{ for a fermion in } |f\rangle & (a^\dagger_{\vec{p},\vec{s}}\, \bar{f}_{\vec{p},\vec{s}}(x) \text{ to match } \langle 0|a_{\vec{p},\vec{s}}\,.) \\
\bar{g}_{\vec{p},\vec{s}}(x) \text{ for an antifermion in } |i\rangle & (b_{\vec{p},\vec{s}}\, \bar{g}_{\vec{p},\vec{s}}(x) \text{ to match } b^\dagger_{\vec{p},\vec{s}}|0\rangle\,.) \\
g_{\vec{p},\vec{s}}(x) \text{ for an antifermion in } |f\rangle & (b^\dagger_{\vec{p},\vec{s}}\, g_{\vec{p},\vec{s}}(x) \text{ to match } \langle 0|b_{\vec{p},\vec{s}}\,.)
\end{array} \tag{5.106}$$

In this list, we note that annilation operators, namely the particles in the initial state, are always associted with a positive energy oscillation $e^{-ip\cdot x}$, and creation operators, namely the particles in the final state, are always associated with a negative energy oscillation $e^{ip\cdot x}$, regardless of the type of particle. Thus, if the initial state momenta are $p_i (i = 1, \ldots, m)$ and the final state momenta are $p_f (f = 1, \ldots, n)$, upon integration over space-time, we will have a delta function corresponding to energy-momentum conservation $(2\pi)^4\delta^4(\sum_i p_i - \sum_f p_f)$. In a general transition $p_i(i = 1, \ldots, m)$ to $p_f(f = 1, \ldots, n)$, the delta function for the energy momentum conservation and the normalization factors $1/\sqrt{2p^0V}$ can be separated out in S_{fi} and the rest defined as the Lorentz-invariant matrix element $\mathcal{M}$:

$$S_{fi} \equiv \frac{(2\pi)^4\delta^4\Big(\sum_i p_i - \sum_f p_f\Big)}{\sqrt{\prod_i(2p_i^0V)\prod_f(2p_f^0V)}}\,\mathcal{M}\,. \tag{5.107}$$

As we will see later, this form holds also when spin-1 particles are involved.

The contributions of the inital and final state particles to the Lorentz-invariant matrix element are then

	initial state	final state
scalar	1	1
fermion	$u_{\vec{p},\vec{s}}$	$\bar{u}_{\vec{p},\vec{s}}$
antifermion	$\bar{v}_{\vec{p},\vec{s}}$	$v_{\vec{p},\vec{s}}$

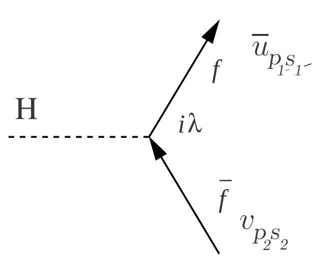
(5.108)

For a n-body decay $P \to p_1, \ldots, p_n$ with masses $M \to m_1, \ldots, m_n$, the same procedure as in the case of 2-body decay can be followed to obtain

$$\begin{aligned} d\Gamma &= \frac{(2\pi)^4}{2P^0} d\Phi_n |\mathcal{M}|^2 \\ d\Phi_n &\stackrel{\text{def}}{\equiv} \delta^4(\sum_f p_f - P) \prod_f \frac{d^3p_f}{(2\pi)^3 2p_f^0} \end{aligned} \tag{5.109}$$

where $d\Phi_n$ is the Lorentz-invariant n-body phase space and p_i^0 is a function of $\vec{p}_i$ as usual: $p_i^0 \equiv \sqrt{\vec{p}_i^{\,2} + m_i^2}$ $(i = 1, \ldots, n)$.

In the case of the decay $H \to f\bar{f}$, the u, v spinors are combined following the form of the interaction Lagrangian $\mathcal{L} = \lambda\phi\bar{\psi}\psi$ with the constant vertex factor given by i (which came from the Dyson series) times whatever the coupling constant is in $\mathcal{L}$. This can be graphically written as below, and one can immediately write down the Lorentz-invariant matrix element:



$$\mathcal{M} = i\lambda\,\bar{u}_{\vec{p}_1,\vec{s}_1} v_{\vec{p}_2,\vec{s}_2}\,. \tag{5.110}$$

The rules (5.108) together with the vertex factor are part of the calculational rules called the Feynman rules, and the diagram is called the Feynman diagram. In the diagram above, time flows from left to right, but this rule is not strictly followed.

Often the arrow of antifermion is reversed as in the figure above to emphasize that the spinor combination $\bar{u}_{\vec{p}_1,\vec{s}_1} v_{\vec{s}_2,\vec{s}_2}$ forms a bilinear covariant (or a general 'current'). In the framework of the non-quantized hole theory, one may interpret that the reversal of the arrow arises because an antifermion corresponds to a negative-energy fermion with momentum and spin that are opposite to those of the physical anti-fermion. In our framework of quantum fields, however, there is no need to resort to such negative energy states. The fermion and antifermion in the final state naturally formed a bilinear covariant because, in the momentum expansion of fields, creation operators of a fermion were associated with $\bar{u}$ spinors and the creation operators of antifermion with v spinors.

Exercise 5.1 *Traces of gamma matrices.*
Let's prove some of the most often used relations in actual calculation of rates. These traces show up when spin average is taken for fermions. Do NOT rely on any specific representation of γ matrices.
(a) *Prove that*

$$\mathrm{Tr}1 = 4\,, \quad \mathrm{Tr}(\not{a}\not{b}) = 4a\cdot b\,. \tag{5.111}$$

(b) *Prove the following trace relation that reduces number of γ matrixes in traces by two:*

$$\begin{aligned}\mathrm{Tr}(\not{a}_1\not{a}_2\cdots\not{a}_{2n}) &= \sum_{i=2}^{2n}(-1)^i a_1\cdot a_i \mathrm{Tr}(\not{a}_2\cdots\not{a}_{i-1}\not{a}_{i+1}\cdots\not{a}_{2n}) \\ &= a_1\cdot a_2\mathrm{Tr}(\not{a}_3\not{a}_4\cdots\not{a}_{2n}) - a_1\cdot a_3\mathrm{Tr}(\not{a}_2\not{a}_4\cdots\not{a}_{2n}) \\ &\quad + \cdots + a_1\cdot a_{2n}\mathrm{Tr}(\not{a}_2\not{a}_3\cdots\not{a}_{2n-1})\,. \end{aligned} \tag{5.112}$$

(*hint: Take the γ matrix on a_1 and shift it all the way over to the right using the anticommutation relation.*)
(c) *Using the result of* (b) *to show that*

$$\mathrm{Tr}(\not{a}\not{b}\not{c}\not{d}) = 4\Big[(a\cdot b)(c\cdot d) - (a\cdot c)(b\cdot d) + (a\cdot d)(b\cdot c)\Big]\,. \tag{5.113}$$

(d) *Prove that the trace of a product of odd number of γ matrices ($\gamma^\mu; \mu = 0,1,2,3$) is zero including the case of single γ matrix:*

$$\mathrm{Tr}(\gamma^{\mu_1}\ldots\gamma^{\mu_{2n+1}}) = 0 \quad (n = 0,1,2\ldots)\,. \tag{5.114}$$

This means that

$$\mathrm{Tr}(\not{a}_1\cdots\not{a}_{2n+1}) = 0 \quad (n = 0,1,2\ldots)\,. \tag{5.115}$$

(*hint: Note first that a trace of odd number of γ's can always written as a trace of three γ's:* $\mathrm{Tr}(\gamma^{\mu_1}\ldots\gamma^{\mu_{2n+1}}) = \mathrm{Tr}(\gamma^\alpha\gamma^\beta\gamma^\gamma)$*. Then there is at least one γ^μ which does not appear. Pick such μ, and write the trace* $\mathrm{Tr}(\gamma^\alpha\gamma^\beta\gamma^\gamma\gamma^\mu\gamma^\mu) = \pm\mathrm{Tr}(\gamma^\alpha\gamma^\beta\gamma^\gamma)$ *in two ways to move one of the γ's to the right hand most; one using* $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$ *and the other by shifting it using anticommutation rule of γ matrices.*)

(e) *Furthermore show that*

$$\mathrm{Tr}\gamma_5 = 0\,, \quad \mathrm{Tr}(\not{a}\not{b}\gamma_5) = 0\,. \tag{5.116}$$

(f) *Prove that*

$$\mathrm{Tr}(\gamma_5\gamma^\alpha\gamma^\beta\gamma^\gamma\gamma^\delta) = 4i\,\epsilon^{\alpha\beta\gamma\delta}\,. \tag{5.117}$$

(*hint: note that* $\mathrm{Tr}(\gamma_5^2) = \mathrm{Tr}1 = 4$*. Replace one of the γ_5's by $i\gamma^0\gamma^1\gamma^2\gamma^3$.*)

(g) *And finally,*

$$\mathrm{Tr}(\gamma^{\mu_1}\gamma^{\mu_2}\ldots\gamma^{\mu_{2n}}) = \mathrm{Tr}(\gamma^{\mu_{2n}}\ldots\gamma^{\mu_2}\gamma^{\mu_1}) \quad (\textit{reverse order})\,. \tag{5.118}$$

Exercise 5.2 *More on γ matrixes.*
Prove the following relations:

$$\begin{aligned}
\gamma^\mu\gamma_\mu &= 4 \\
\gamma^\mu\gamma^\alpha\gamma_\mu &= -2\gamma^\alpha \\
\gamma^\mu\gamma^\alpha\gamma^\beta\gamma_\mu &= 4g^{\alpha\beta} \\
\gamma^\mu\gamma^\alpha\gamma^\beta\gamma^\delta\gamma_\mu &= -2\gamma^\delta\gamma^\beta\gamma^\alpha\,.
\end{aligned} \tag{5.119}$$

[*comment: These relations lead to*

$$\begin{aligned}
\gamma^\mu\not{a}\gamma_\mu &= -2\not{a} \\
\gamma^\mu\not{a}\not{b}\gamma_\mu &= 4a\cdot b \\
\gamma^\mu\not{a}\not{b}\not{c}\gamma_\mu &= -2\not{c}\not{b}\not{a}\,.
\end{aligned} \tag{5.120}$$

Note that the index μ is summed in all of the above.]

5.4 Muon decay

The muon decay $\mu^- \to \nu_\mu e^-\bar{\nu}_e$, which proceeds with essentially 100% branching fraction, is an example of weak interaction which is mediated by W and Z vector bosons. All particles involved in the muon decay are fermions: μ^- is a charged fermion of mass $m = 0.106$ GeV, and ν_μ and $\bar{\nu}_e$ are the muon nutrino and electron antineutrino both assumed to be massless. The mass of the electron (~ 0.0005 GeV) is much smaller than that of the muon, and thus we will ignore the electron mass in the following calculation. For now we will assume that the interaction Lagrangian is given by

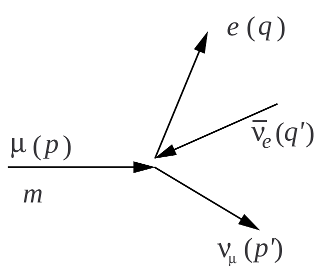
$$\mathcal{L}_{\text{int}} = \frac{G_F}{\sqrt{2}}\Big(\bar{\psi}_{\nu_\mu}\gamma_\alpha(1-\gamma_5)\psi_\mu\Big)\Big(\bar{\psi}_e\gamma^\alpha(1-\gamma_5)\psi_{\nu_e}\Big) + h.c. \tag{5.121}$$

where ψ_μ is the muon field, ψ_e is the electron field, ψ_{ν_μ} and ψ_{ν_e} are the neutrino fields. The Lorentz index α is contracted. This form was first suggested by Fermi, and the constant G_F is called the Fermi coupling constant. Since $\dim(\mathcal{L}) = E^4$ and $\dim(\psi) = E^{3/2}$ (see p197), the dimension of the coupling constant should be $\dim(G_F) = E^{-2}$. This is actually a low-energy approximation of a more fundamental interaction involving W vector boson that we will study later. In fact, we will find that the Fermi coupling constant is given by

$$G_F = \frac{g^2}{4\sqrt{2}\,m_W^2}\,, \tag{5.122}$$

where g is the dimensionless universal coupling constant and m_W is the W boson mass, the same quantities we have seen in (5.104). Here, we will take the measured muon lifetime as an input and extract the value of G_F.

We could simply apply the Feynman rules (5.108), but let us review the derivation from the first principles. The first-order S_{fi} is given by



$$S_{fi} = i\frac{G_F}{\sqrt{2}}\int d^4x\,\langle f|\,\bar{\psi}_{\nu_\mu}\gamma_\alpha(1-\gamma_5)\psi_\mu \times \bar{\psi}_e\gamma^\alpha(1-\gamma_5)\psi_{\nu_e}\,|i\rangle\,. \tag{5.123}$$

We did not use the hermitian-conjugate part of (5.121) since the term we took contains annihilation operator for μ^-; the hermitian-conjugate part would be responsible for the decay of μ^+. Upon momentum-expanding the fields, only the term that annihilates the initial-state muon and creates the final-state particles will survive. The integration over x will give the delta-function for the energy-momentum conservation, and we obtain

$$S_{fi} = \frac{(2\pi)^4\delta^4(p'+q+q'-p)}{\sqrt{(2p^0V)(2p'^0V)(2q^0V)(2q'^0V)}}\mathcal{M} \tag{5.124}$$

with

$$\mathcal{M} = i\frac{G_F}{\sqrt{2}}\,(\overline{u}_{\nu_\mu}\gamma_\alpha(1-\gamma_5)u_\mu)(\overline{u}_e\gamma^\alpha(1-\gamma_5)v_{\nu_e}), \tag{5.125}$$

which could also be directly obtained from the Feynman rules where it is understood that the u, v spinors are combined with the γ matrices to form currents corresponding to those in $\mathcal{L}_{\rm int}$. Summing over all final states with $\vec{p}\,' \in d^3p'$, $\vec{q} \in d^3q$, and $\vec{q}\,' \in d^3q'$, the corresponding decay rate is [see (5.109)]

$$d\Gamma = \frac{(2\pi)^4}{2p^0} d\Phi_3 |\mathcal{M}|^2 \tag{5.126}$$

where the Lorentz-invariant 3-body phase space is given by

$$d\Phi_3 \equiv \delta^4(p' + q + q' - p) \frac{d^3p'}{(2\pi)^3 2p'^0} \frac{d^3q}{(2\pi)^3 2q^0} \frac{d^3q'}{(2\pi)^3 2q'^0} \,. \tag{5.127}$$

Assuming that the parent particle is unpolarized, the 3-body phase space can be integrated over all variables except for E_1 and E_2, which are the energies of *any* two of the three final-state particles in the C.M. frame of the parent, to obtain (left as an exercise)

$$\boxed{d\Phi_3 = \frac{dE_1 dE_2}{4(2\pi)^7}}\,, \tag{5.128}$$

which, together with (5.126), gives the differential decay rate in the C.M frame:

$$\boxed{d\Gamma = \frac{dE_1 dE_2}{64\pi^3\, m} |\mathcal{M}|^2} \quad \text{(3-body, parent unpolarized)} \tag{5.129}$$

where m is the mass of the parent ($p^0 = m$ in the C.M. frame). This formula gives the decay rate into an area element $dE_1 dE_2$ where E_1 and E_2 are the energies of any two of the three daughters. The probability density of the decay in the 2-dimensional space of E_1 vs E_2 is proportional to $|\mathcal{M}|^2$, and such density plot, called the *Dalitz plot*, provides a powerful tool to study the decay matrix element.

Exercise 5.3 *Three-body phase space.*
(*a*) *Take the expression of the 3-body Lorentz-invariant phase space (5.127) and show that it reduces to (5.128) when evaluated in the C.M. system ($E_i \equiv p_i^0$). The particles 1,2,3 have masses m_1, m_2, m_3 and 4-momenta p_1, p_2, p_3, respectively. Assume that the parent particle is spinless (or unpolarized) thus there is no special direction in the C.M. frame; namely, the matrix element does not depend on the direction of the first particle you pick. The matrix element, however, is in general a function of the angle between 1 and 2 among other variables. [hint: Namely, you can set $d^3p_1 = 4\pi \breve{p}_1^2 d\breve{p}$, but then, you cannot do the same for $\vec{p}_2$; it should be $d^3p_2 = 2\pi \breve{p}_2^2 d\breve{p}_2 d\cos\theta_{12}$ ($\breve{p}_i \equiv |\vec{p}_i|$).]*
(*b*) *Show that invariant masses of 2,3 and that of 3,1 are linearly related to E_1 and E_2 by*

$$E_1 = \frac{M^2 + m_1^2 - S_{23}}{2M}\,, \quad E_2 = \frac{M^2 + m_2^2 - S_{31}}{2M}\,,$$

where $S_{ij} = (p_i + p_j)^2$ and M is the mass of the parent particle (or the invariant mass of the entire system in the case of scattering). Then rewrite $d\Phi_3$ in terms of S_{23} and S_{31}. Since S_{23} and S_{31} are Lorentz-invariant quantities, the resulting expression is valid in any frame. [hint: Energy-momentum conservation gives $P - p_1 = p_{12}$ with $p_{12} = p_1 + p_2$. Square both sides and use $P = (M, \vec{0})$.]

The Lorentz-invariant matrix element given in (5.125) is for specific spin polarizations of the initial- and final-state particles. In calculating the decay rate, we sum $|\mathcal{M}|^2$ over all possible spins in the final state. Also, we assume that the parent is unpolarized, and thus we take the average over two possible spin states of the parent. Namely, we sum up all the spins of all initial and final state particles and then divide by two. The result is the 'spin-averaged' matrix element squared - denoted as $\overline{|\mathcal{M}|^2}$:

$$\begin{aligned}
\overline{|\mathcal{M}|^2} &= \frac{1}{2}\sum_{\text{spins}} |\mathcal{M}|^2 \\
&= \frac{1}{2}\frac{G_F^2}{2}\sum_{\text{spins}} (\overline{u}_{\nu_\mu}\gamma_\alpha(1-\gamma_5)u_\mu)^*(\overline{u}_e\gamma^\alpha(1-\gamma_5)v_{\nu_e})^* \\
&\qquad\qquad \times (\overline{u}_{\nu_\mu}\gamma_\beta(1-\gamma_5)u_\mu)(\overline{u}_e\gamma^\beta(1-\gamma_5)v_{\nu_e}) \\
&= \frac{G_F^2}{4}\sum_{\substack{\mu,\nu_\mu \\ \text{spins}}} (\overline{u}_{\nu_\mu}\gamma_\alpha(1-\gamma_5)u_\mu)^*(\overline{u}_{\nu_\mu}\gamma_\beta(1-\gamma_5)u_\mu) \\
&\qquad \times \sum_{\substack{e,\nu_e \\ \text{spins}}} (\overline{u}_e\gamma^\alpha(1-\gamma_5)v_{\nu_e})^*(\overline{u}_e\gamma^\beta(1-\gamma_5)v_{\nu_e})\,. \qquad (5.130)
\end{aligned}$$

Noting that $\gamma^\alpha(1-\gamma_5)$ is self-adjoint:

$$\overline{\gamma^\alpha(1-\gamma_5)} = (1 - \underbrace{\overline{\gamma_5}}_{-\gamma_5})\underbrace{\overline{\gamma^\alpha}}_{\gamma^\alpha} = (1+\gamma_5)\gamma^\alpha = \gamma^\alpha(1-\gamma_5)\,, \qquad (5.131)$$

and using the spin-sum formulas $\sum_{\text{spin}} u_{\nu_\mu}\overline{u}_{\nu_\mu} = \not{p}' + \cancel{m}_{\nu_\mu}$ $(m_{\nu_\mu} = 0)$ etc. as well as (5.100), the spin average leads to traces:

$$\begin{aligned}
\overline{|\mathcal{M}|^2} &= \frac{G_F^2}{4}\sum_{\substack{\mu,\nu_\mu \\ \text{spins}}} \overline{u}_\mu\gamma_\alpha(1-\gamma_5)\underbrace{u_{\nu_\mu}\overline{u}_{\nu_\mu}}_{\to\,\not{p}'}\gamma_\beta(1-\gamma_5)u_\mu \\
&\qquad \times \sum_{\substack{e,\nu_e \\ \text{spins}}} \overline{v}_{\nu_e}\gamma^\alpha(1-\gamma_5)\underbrace{u_e\overline{u}_e}_{\to\,\not{q}}\gamma^\beta(1-\gamma_5)v_{\nu_e} \\
&= \frac{G_F^2}{4}\mathrm{Tr}\,(\not{p}+m)\gamma_\alpha(1-\gamma_5)\not{p}'\gamma_\beta(1-\gamma_5) \\
&\qquad \times\mathrm{Tr}\,\not{q}'\gamma^\alpha(1-\gamma_5)\not{q}\gamma^\beta(1-\gamma_5) \qquad (5.132)
\end{aligned}$$

Since $\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3$ contains four γ^μ's, the term linear in m in the first trace vanishes becuase of the odd number of γ^μ's in the trace. Noting that γ_5 commutes with $\not{p}'\gamma_\beta$, we have

$$\mathrm{Tr}\,(\not{p} + \cancel{m})\gamma_\alpha \underbrace{(1-\gamma_5)\not{p}'\gamma_\beta}_{\not{p}'\gamma_\beta(1-\gamma_5)}(1-\gamma_5) = \mathrm{Tr}\not{p}\gamma_\alpha\not{p}'\gamma_\beta \underbrace{(1-\gamma_5)^2}_{2(1-\gamma_5)} \,. \tag{5.133}$$

Similarly,

$$\mathrm{Tr}\,\not{q}\gamma^\alpha(1-\gamma_5)\not{q}'\gamma^\beta(1-\gamma_5) = 2\mathrm{Tr}\not{q}'\gamma^\alpha\not{q}\gamma^\beta(1-\gamma_5)\,. \tag{5.134}$$

Thus,

$$\overline{|\mathcal{M}|^2} = G_F^2\mathrm{Tr}\not{p}\gamma_\alpha\not{p}'\gamma_\beta(1-\gamma_5) \times \mathrm{Tr}\not{q}'\gamma^\alpha\not{q}'\gamma^\beta(1-\gamma_5)\,. \tag{5.135}$$

Using the trace theorems

$$\mathrm{Tr}\gamma_\mu\gamma_\alpha\gamma_\nu\gamma_\beta = 4(g_{\mu\alpha}g_{\nu\beta} - g_{\mu\nu}g_{\alpha\beta} + g_{\mu\beta}g_{\nu\alpha}) \tag{5.136}$$

$$\mathrm{Tr}\gamma_5\gamma_\mu\gamma_\alpha\gamma_\nu\gamma_\beta = 4i\epsilon_{\mu\alpha\nu\beta}\,, \tag{5.137}$$

we have

$$\begin{aligned}
&\mathrm{Tr}\not{p}\gamma_\alpha\not{p}'\gamma_\beta(1-\gamma_5) \\
&\quad = p^\mu p'^\nu(\mathrm{Tr}\gamma_\mu\gamma_\alpha\gamma_\nu\gamma_\beta - \mathrm{Tr}\gamma_5\gamma_\mu\gamma_\alpha\gamma_\nu\gamma_\beta) \\
&\quad = 4p^\mu p'^\nu(g_{\mu\alpha}g_{\nu\beta} - g_{\mu\nu}g_{\alpha\beta} + g_{\mu\beta}g_{\nu\alpha} - i\epsilon_{\mu\alpha\nu\beta})\,.
\end{aligned} \tag{5.138}$$

Applying the same procedure to the second trace also, we obtain

$$\mathrm{Tr}\not{p}\gamma_\alpha\not{p}'\gamma_\beta(1-\gamma_5) = 4(p_\alpha p'_\beta - p\cdot p' g_{\alpha\beta} + p_\beta p'_\alpha - ip^\mu p'^\nu\epsilon_{\mu\alpha\nu\beta}) \tag{5.139}$$

$$\mathrm{Tr}\not{q}'\gamma^\alpha\not{q}\gamma^\beta(1-\gamma_5) = 4(q'^\alpha q^\beta - q'\cdot q\, g^{\alpha\beta} + q'^\beta q^\alpha - iq'_\rho q_\sigma\epsilon^{\rho\alpha\sigma\beta})\,. \tag{5.140}$$

In each of the traces, the first three terms are symmetric under the exchange $\alpha \leftrightarrow \beta$ while the last term is antisymmetric. Thus, when we take the product of the two traces, the cross terms between symmetric and antisymmetric terms vanish; thus,

$$\begin{aligned}
\overline{|\mathcal{M}|^2} = 16G_F^2\Big[&(p_\alpha p'_\beta - p\cdot p' g_{\alpha\beta} + p_\beta p'_\alpha)(q'^\alpha q^\beta - q'\cdot q\, g^{\alpha\beta} + q'^\beta q^\alpha) \\
&-p^\mu p'^\nu q'_\rho q_\sigma\epsilon_{\mu\alpha\nu\beta}\epsilon^{\rho\alpha\sigma\beta}\Big]\,.
\end{aligned} \tag{5.141}$$

The first term is

$$\begin{aligned}
&\Big[(p_\alpha p'_\beta + p_\beta p'_\alpha) - p\cdot p' g_{\alpha\beta}\Big]\Big[(q'^\alpha q^\beta + q'^\beta q^\alpha) - q'\cdot q\, g^{\alpha\beta}\Big] \\
&\quad = \underbrace{(p_\alpha p'_\beta + p_\beta p'_\alpha)(q'^\alpha q^\beta + q'^\beta q^\alpha)}_{2p\cdot q'\, p'\cdot q + 2p\cdot q\, p'\cdot q'} - 4p\cdot \cancel{p'q'}\cdot q + p\cdot\cancel{p'q'}\cdot q\, \underbrace{g_{\alpha\beta}g^{\alpha\beta}}_{4} \\
&\quad = 2(p\cdot q'\, p'\cdot q + p\cdot q\, p'\cdot q')\,,
\end{aligned} \tag{5.142}$$

while the second term uses an identity for a product of two $\epsilon_{\mu\nu\alpha\beta}$'s:

$$-p^\mu p'^\nu q'_\rho q_\sigma \underbrace{\epsilon_{\mu\alpha\nu\beta}\epsilon^{\rho\alpha\sigma\beta}}_{\epsilon_{\mu\nu\alpha\beta}\epsilon^{\rho\sigma\alpha\beta} = -2(g^\rho_\mu g^\sigma_\nu - g^\sigma_\mu g^\rho_\nu)} = 2(p\cdot q'\, p'\cdot q - p\cdot q\, p'\cdot q')\,. \tag{5.143}$$

Finally, we obtain

$$\overline{|\mathcal{M}|^2} = 64 G_F^2\, p\cdot q'\, p'\cdot q\,. \tag{5.144}$$

Using 4-momentum conservation $p = p' + q + q'$, we can relate $p'\cdot q$ and $p\cdot q'$:

$$(p-q')^2 = (p'+q)^2 \;\to\; \underbrace{p^2}_{m^2} + \underbrace{q'^2}_{0} - 2p\cdot q' = \underbrace{p'^2}_{0} + \underbrace{q^2}_{0} + 2p'\cdot q$$
$$\to\; p'\cdot q = \frac{m^2}{2} - p\cdot q'\,. \tag{5.145}$$

In the C.M. frame of the parent, we have

$$p = (m,\vec{0}\,) \quad\to\quad p\cdot q' = mE_{\bar\nu_e} \quad \text{(C.M. frame)}\,; \tag{5.146}$$

thus, the spin-averaged $|\mathcal{M}|^2$ can be written as

$$\overline{|\mathcal{M}|^2} = 64 G_F^2\, mE_{\bar\nu_e}\Big(\frac{m^2}{2} - mE_{\bar\nu_e}\Big)\,. \tag{5.147}$$

We can now use the 3-body decay rate formula (5.129) to obtain the differential decay rate. The formula is valid for any two energies, E_1 and E_2, of the final state as long as the parent is unpolarized. It is reassuring that indeed the expression of $\overline{|\mathcal{M}|^2}$ obtained above does not depend on spacial direction: it depends only on the energy of the electron antineutrino. It is then natural to take one of the energies to be $E_{\bar\nu_e}$. What shall we take for the other energy? Actually, it does not matter; we could take E_{ν_μ} or E_e and both should give the correct distribution. This means that e^- and ν_μ have exactly the same energy distribution (in the limit of $m_e = 0$). Let's take $E_2 = E_e$. Then, we obtain

$$d\Gamma = \frac{G_F^2 m}{\pi^3} E_{\bar\nu_e}\Big(\frac{m}{2} - E_{\bar\nu_e}\Big) dE_{\bar\nu_e} dE_e\,. \tag{5.148}$$

Now, when a particle of mass m at rest decays to three massless particles, the maximum energy of any of the daughters is $m/2$, which can be seen as follows: first, $p = p_1 + p_2 + p_3$ gives

$$\underbrace{(p-p_1)^2}_{m^2 - 2mE_1} = (p_2+p_3)^2 \equiv s_{23} \quad\to\quad E_1 = \frac{m^2 - s_{23}}{2m}\,, \tag{5.149}$$

where the invariant mass squared of the particle 2 and 3, s_{23}, is given by

$$s_{23} = \underbrace{p_2^2 + p_3^2}_{0} + 2p_2p_3 = 2E_2E_3(1 - \cos\theta_{23}) . \tag{5.150}$$

Namely, the energy of a given daughter (E_1) is larger when the invariant mass squared of the system recoiling against it (s_{23}) is smaller, and s_{23} is minimum, in fact zero, when the angle between the particles 2 and 3 is zero regardless of E_2 and E_3. Setting $s_{23} = 0$ in (5.149) the maximum energy is $E_1 = m/2$. Define the dimensionless parameters x_i $(i = \bar{\nu}_e, e, \nu_\mu)$ by

$$E_i \equiv \frac{m}{2} x_i , \qquad 0 \le x_i \le 1 \quad (i = \bar{\nu}_e, e, \nu_\mu) . \tag{5.151}$$

Then, the decay rate is written as

$$d\Gamma = \frac{G_F^2 m^5}{16\pi^3} x_{\bar{\nu}_e}(1 - x_{\bar{\nu}_e}) dx_{\bar{\nu}_e} dx_e , \tag{5.152}$$

which gives the probability distribution in the plane of $x_{\bar{\nu}_e}$ vs x_e (Figure 5.2). Not all points in $0 \le x_{\bar{\nu}_e} \le 1$ and $0 \le x_e \le 1$ are kinematically allowed. When $x_{\bar{\nu}_e} = x_e = 0$, for example, all energy has to be carried by ν_μ to conserve energy, but it cannot have any energy without violating momentum conservation. Energy conservation gives the allowed region:

$$\begin{aligned} E_{\bar{\nu}_e} + E_e + E_{\nu_\mu} = m \ &\to \ x_{\bar{\nu}_e} + x_e + x_{\nu_\mu} = 2 , \\ &\to \ x_{\bar{\nu}_e} + x_e = 2 - \underbrace{x_{\nu_\mu}}_{\le 1} \\ &\to \ x_{\bar{\nu}_e} + x_e \ge 1 \end{aligned} \tag{5.153}$$

which is shown in Figure 5.2. Taking only the allowed region, and projecting it on to $x_{\bar{\nu}_e}$ and x_e axes, we obtain the energy distributions for $\bar{\nu}_e$ and e^-, respectively. Since ν_μ should have the same energy distribution as e^- as discussed earlier, we see that fermions (e^- and ν_μ) have a distribution that peaks at the maximum energy, and the antifermion ($\bar{\nu}_e$) has a distribution that peaks near the center.

The total decay rate is obtained by integrating (5.152) inside the allowed region. Figure 5.2 shows that the integral in the the allowed region is the same as that in the non-allowed region; thus,

$$\begin{aligned} \Gamma &= \int_{x_{\bar{\nu}_e}+x_e>1} d\Gamma = \int_{x_{\bar{\nu}_e}+x_e<1} d\Gamma = \frac{1}{2}\int d\Gamma \\ &= \frac{G_F^2 m^5}{32\pi^3} \underbrace{\int_0^1 dx_e}_{1} \underbrace{\int_0^1 dx_{\bar{\nu}_e} x_{\bar{\nu}_e}(1 - x_{\bar{\nu}_e})}_{1/6} ; \end{aligned} \tag{5.154}$$

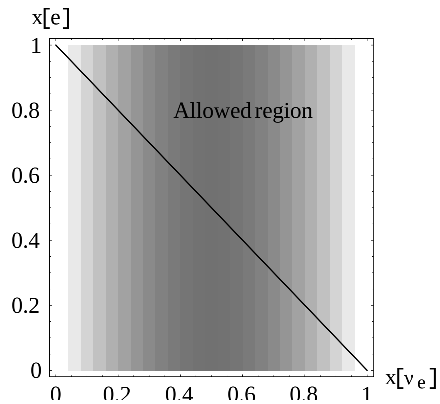


Figure 5.2: The probability density distribution (the Dalitz plot) of the decay $\mu^- \to \nu_\mu e^- \bar{\nu}_e$. The density is proportional to the decay probability. The kinematically allowed region is $x_{\bar{\nu}_e} + x_e \geq 1$. The plot on the right shows the projections of the allowed region on to $x_{\bar{\nu}_e}$ (solid line) and x_e (dashed line).

thus, the total decay rate is

$$\Gamma(\mu^- \to \nu_\mu e^- \bar{\nu}_e) = \frac{G_F^2 m^5}{192\pi^3} \,. \tag{5.155}$$

The experimental lifetime τ_μ can be converted to the decay rate:

$$\begin{aligned} \tau_\mu &= 2.20 \times 10^{-6} \text{ (sec)} \\ \to \quad \Gamma &= \frac{\hbar}{\tau_\mu} = \frac{6.5822 \times 10^{-25} \text{ (GeV}\cdot\text{sec)}}{2.20 \times 10^{-6} \text{ (sec)}} = 2.99 \times 10^{-19} \text{ (GeV)} \ . \end{aligned} \tag{5.156}$$

Together with $m = 0.106$ GeV, we then obtain $G_F = 1.15 \times 10^{-5}$ GeV^{-2}. In order to obtain a more accurate value, one needs to include higher-order processes where photons are emitted from or absorbed by charged particles. The correction amounts to a 0.42% reduction in the theoretical decay rate. The most up-to-date value is

$$G_F = 1.16639(2) \times 10^{-5} \text{ (GeV}^{-2}\text{)} \,. \tag{5.157}$$

Spin polarizations

The interaction (5.121) is a product of two currents each of which has the form a vector current minus an axial vector current:

$$\bar{\psi}'\gamma^\alpha(1-\gamma_5)\psi = \underbrace{\bar{\psi}'\gamma^\alpha\psi}_{V} - \underbrace{\bar{\psi}'\gamma^\alpha\gamma_5\psi}_{A} \,. \tag{5.158}$$

This $V-A$ form of the interaction and the resulting Lorentz-invariant matrix element (5.125) tells us how the final-state particles are polarized. Let's write $\mathcal{M}$ as

$$\mathcal{M} \propto (\overline{u}_{\nu_\mu}\gamma_\alpha P_L u_\mu)(\overline{u}_e\gamma^\alpha P_L v_{\nu_e})\,, \tag{5.159}$$

where $P_L \equiv (1-\gamma_5)/2$ was defined in (3.365), where we have seen that P_L (P_R) acts as the helicity $-\,(+)$ projection operator for fermions and as the helicity $+\,(-)$ projection operator for antifermions in the massless limit or, equivalently, in the high-energy limit.

Suppose the spinor v_{ν_e} represents an antineutrino with negative helicity. Since P_R acts as the negative helicity projection operator for antifermions, v_{ν_e} should satisfy

$$v_{\nu_e} = P_R\, v_{\nu_e} \quad (\bar{\nu}_e \text{ helicity } -), \tag{5.160}$$

then $\mathcal{M}$ vanishes due to $P_L P_R = 0$ (3.347):

$$(\overline{u}_e\gamma^\alpha P_L \underbrace{v_{\nu_e}}_{P_R\, v_{\nu_e}}) = (\overline{u}_e\gamma^\alpha \underbrace{P_L P_R}_{0} v_{\nu_e}) = 0\,. \tag{5.161}$$

This means that $\bar{\nu}_e$ cannot be created with negative helicity. Since the spin of $\bar{\nu}_e$ should be either helicity $+$ or $-$, the vanishing amplitude for negative helicity indicates that $\bar{\nu}_e$ in the muon decay is in a pure helicity $+$ state. Now, suppose the spinor u_e represents an electron with positive helicity. Then, to the extent we can ignore the mass of electron,

$$u_e = P_R\, u_e \quad (e^- \text{ helicity } +). \tag{5.162}$$

Then, using $\bar{u}_e = \overline{P_R u_e} = \bar{u}_e \overline{P}_R = \bar{u}_e P_L$ and $P_L\gamma^\alpha = \gamma^\alpha P_R$, we see that $\mathcal{M}$ vanishes again:

$$(\underbrace{\overline{u}_e}_{\bar{u}_e P_L}\gamma^\alpha P_L v_{\nu_e}) = (\overline{u}_e \underbrace{P_L\gamma^\alpha P_L}_{\gamma^\alpha P_R P_L = 0} v_{\nu_e}) = 0\,. \tag{5.163}$$

Namely, the electron is created with negative helicity. Similarly, we see that the muon neutrino is purely left-handed. Thus, a massless fermion created by a $V-A$ current is always left-handed (i.e. negative helicity), and a massless antifermion created by a $V-A$ current is always right-handed (i.e. positive helicity).

One can use the helicity projection operator to calculate the decay rate where a given (massless) fermion has particular helicity. As an example, let's take the Higgs decay $H \to f\bar{f}$ where the mass of the fermion is small compared to its momentum, and calculate the decay rate where the antifermion, represented by $v_2 \equiv v_{\vec{p}_2,\vec{s}_2}$, is left-handed. We can still take advantage of the trace technique in the following way: First, assume that the spin quantization axis is taken as $\vec{s} = \hat{p}$ such that the two possible polarizations correspond to positive and negative helicities. In evaluating the spin sum of $|\mathcal{M}|^2 = \lambda^2|\bar{u}_1 v_2|^2$ in (5.99), we can place P_R in front of v_2 and then sum over the spins of f and $\bar{f}$. Then the matrix element should vanish unless the v_2 represents a left-handed antifermion, and the resulting decay rate should correspond to the case where the antifermion is left-handed. With $u_1 \equiv u_{\vec{p}_1,\vec{s}_1}$ and ignoring the fermion mass, we have

$$\begin{aligned}
&\sum_{\text{spins}} |\mathcal{M}|^2 \;\; (\bar{f} \text{ left-handed}) \\
&\qquad = \lambda^2 \sum_{\text{spins}} |\bar{u}_1 P_R\, v_2|^2 = \lambda^2 \sum_{\text{spins}} \bar{v}_2 P_L \underbrace{u_1\bar{u}_1}_{\to\, \not{p}_1} P_R\, v_2 \\
&\qquad = \lambda^2 \mathrm{Tr} \not{p}_2 \underbrace{P_L \not{p}_1}_{\not{p}_1 P_R} P_R = \lambda^2 \mathrm{Tr} \not{p}_2 \not{p}_1 \underbrace{P_R^2}_{P_R} \\
&\qquad = \lambda^2 \mathrm{Tr} \not{p}_2 \not{p}_1 \frac{1+\gamma_5}{2} = \frac{\lambda^2}{2}\Big(\mathrm{Tr} \not{p}_2 \not{p}_1 + \underbrace{\mathrm{Tr} \not{p}_2 \not{p}_1 \gamma_5}_{0} \Big) \\
&\qquad = \frac{\lambda^2}{2} \mathrm{Tr} \not{p}_2 \not{p}_1 \,.
\end{aligned} \tag{5.164}$$

This is exactly one half of the sum without any helicity restriction (5.99) with $m = 0$. Thus, we see that $\bar{f}$ is left-handed 50% of the time. Similarly, one can see that it is right-handed 50% of the time. In addition, a technique similar to the muon decay case immediately tells us that the two daughters have to be both left-handed or both right-handed. For example, if f is left-handed and $\bar{f}$ is right-handed, we can replace u_1 by $P_L u_1$ and v_2 by $P_L v_2$, and the matrix element becomes

$$\mathcal{M} \propto \bar{u}_1 v_2 = \overline{P_L u_1} P_L v_2 = \bar{u}_1 P_R P_L v_2 = 0 \,. \tag{5.165}$$

Thus, each daughter is unpolarized when viewed individually, but there is a correlation between the polarizations of the two.

How can we calculate the rate where a fermion is polarized in some arbitrary direction, or if the fermion is heavy, for that matter? Fortunately, we have a spin projection operator for massive fermion or antifermion along any direction. All we need is then to place the spin projection operator (3.277) in front of the spinor in question

$$w_{\vec{p},\pm\vec{s}} \quad \to \quad \frac{1 \pm \gamma_5 \not{s}}{2} w_{\vec{p},\pm\vec{s}} \,, \qquad (w_{\vec{p},\vec{s}} = u_{\vec{p},\vec{s}} \text{ or } v_{\vec{p},\vec{s}}) \tag{5.166}$$

and then execute the spin sum, where s^μ is the boosted unit vector $(0, \vec{s})$ which defines the spin quantization axis in the rest frame of the particle, and the $\pm$ sign corresponds to the physical spin component along $\vec{s}$ in the rest frame of the particle. Note that the same projection operator works for both fermion and antifermion. In fact, one can insert the spin projection operator for f or $\bar{f}$ in the evaluation of $H \to f\bar{f}$ and show that the rate is always 1/2 of the total if the spin of f or $\bar{f}$ is restricted to one polarization, and this is so regardless of the spin direction and even when the daughters are massive. Namely, the fermion and the antifermion in $H \to f\bar{f}$ are unpolarized when examined individually regardless of the fermion mass.

5.5 Spin-1 Fields

We will now introduce spin-1 fields. Examples are the vector bosons $W^\pm$ and Z which mediate the weak interaction, and in the massless limit we have photon which mediates the electro-magnetic interaction. These are so-called *gauge bosons* which are responsible for gauge interactions between fermions and sometimes called force particles. A pair of fermions can form a spin-1 bound state (as well as other integer spins), and this section applies to those bound-state particles also. As we will see, the simple massless limit for spin-1 particle encounters difficulties, and the rigorous treatment requires an understanding of a symmetry introduced by the masslessness called the *gauge invariance* which will be discussed in later chapters. We start from the Lagrangian formulation of free massive spin-1 field.

Lagrangian for a free massive spin-1 field

Let's start from searching for non-quantized fields that can represent a spin-1 particle. We have seen that single component real field can represent a single spin-0 particle which is reasonable since a spin-0 particle at rest has only one degree of freedom. A spin-1 particle at rest, on the other hand, have three degrees of freedom:

$$|jm\rangle = |1,+1\rangle\,, \quad |1,0\rangle\,, \quad |1,-1\rangle\,, \tag{5.167}$$

where j is the absolute value of the spin and m is the component along some axis. Then, we expect that we need three components of real field to represent it. Also, the three components have to transform under rotation in a way consistent with a spin-1 particle. We recall that under a rotation the Dirac field transformed as

$$\psi'(x') = U\psi(x) \qquad U = e^{-i\frac{\Sigma_i}{2}\theta_i}\,, \tag{5.168}$$

and the *generator* $\vec{J} = \vec{\sigma}/2$ satisfied the commutation relations of angular momentum $[J_i, J_j] = i\epsilon_{ijk}J_k$ and $\vec{J}^2 = j(j+1)$ with $j = 1/2$ (3.158). In our case, we need three components that transform under rotation by $e^{-i\vec{\theta}\cdot\vec{J}}$ where $[J_i, J_j] = i\epsilon_{ijk}J_k$ and $\vec{J}^2 = j(j+1) = 1(1+1) = 2$. Actually we already have such quantity - the space components of a 4-vector. In fact, we have seen that a rotation in the ordinary three-dimensional space can be written as $e^{-i\vec{\theta}\cdot\vec{J}}$ with $(J_i)_{jk} = i(L_i)^j{}_k$ (3.160):

$$J_1 = i\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}\,, \quad J_2 = i\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}\,, \quad J_3 = i\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\,, \tag{5.169}$$

and that $\vec{J}$ defined as such satisfy the commutation relations of angular momentum operator. Recalling that $(L_i)^j{}_k = -\epsilon_{ijk}$ (1.101), the components $(J_i)_{jk}$ is nothing but $(-i)$ times the structure consant ϵ_{ijk}:

$$(J_i)_{jk} = -i\epsilon_{ijk}\,. \tag{5.170}$$

It is a general feature of Lie algebra that a representation of generators can be consturcted directly out of the structure constants, and such representation is called the *adjoint representation*. For the absolute value of the spin, explicit evaluation using (5.169) indeed shows that it is spin one:

$$\vec{J}^2 = 2I \quad \to \quad j = 1\,. \tag{5.171}$$

Thus, in order to represent a spin-1 particle, we take a *real* 4-vector field which transforms under Lorentz transformation as

$$\boxed{A'^\mu(x') = \Lambda^\mu{}_\nu A^\nu(x)} \quad (x' = \Lambda x)\,. \tag{5.172}$$

There are, however, four degrees of freedom for A^μ $(\mu = 0, 1, 2, 3)$ while we need only three. We will try to remove the one extra degree of freedom by imposing the condition

$$\boxed{\partial_\mu A^\mu(x) = 0}\,. \tag{5.173}$$

Let's try to see what this means by applying it to plane waves

$$A^\mu(x) \propto e^{\pm ip\cdot x}\,. \tag{5.174}$$

The condition $\partial_\mu A^\mu(x) = 0$ is then written as

$$p_\mu A^\mu(x) = 0\,, \tag{5.175}$$

which reads in the rest frame of the particle as

$$p = (m, \vec{0}\,) \quad \to \quad mA^0(x) = 0 \quad \to \quad A^0(x) = 0 \text{ (if } m \neq 0, \text{ at rest)}\,. \tag{5.176}$$

Thus, the condition $\partial_\mu A^\mu(x) = 0$ removes one degree of freedom out of the 4-vector such that for a particle at rest the time component A^0 is zero while keeping the 4-vector nature of the field intact. The condition $\partial_\mu A^\mu(x) = 0$ is called the *Lorentz condition* which is a transversality condition in the four-dimensional space-time.

How about the equation of motion and the Lagrangian density that leads to it? In order to be consistent with the relativistic energy-momentum relation ${p^0}^2 = \vec{p}^{\,2} + m^2$, each plane wave solution, and thus any free-field solution, should satisfy the Klein-Gordon equation

$$(\partial^2 + m^2)A^\mu(x) = 0 \qquad (\mu = 0, 1, 2, 3)\,. \tag{5.177}$$

If we can regard each component as independent, then the total Lagrangian density would be simply the sum of the Klein-Gordon Lagrangian density for each component:

$$\mathcal{L} \stackrel{?}{=} -\frac{1}{2}(\partial_\nu A_\mu \partial^\nu A^\mu + m^2 A_\mu A^\mu)\,, \tag{5.178}$$

where we were 'forced' to flip the sign for the Lagrangian density of the time component $A^0(x)$ in order to make the whole a Lorentz scalar. One problem, of course, is that this has four dynamical degrees of freedom instead of three. One way to remove one dynamical degree of freedom is to eliminate the time derivative of one component so that the corresponding conjugate field vanishes. This can be accomplished in a Lorentz-invariant way by forming the antisymmetric combination

$$\boxed{F^{\mu\nu} \stackrel{\text{def}}{\equiv} \partial^\nu A^\mu - \partial^\mu A^\nu\,, \quad F^{\mu\nu} = -F^{\nu\mu}} \tag{5.179}$$

which has no $\dot{A}^0$ in it:

$$F^{00} = \partial^0 A^0 - \partial^0 A^0 = 0\,, \tag{5.180}$$

and try the following form

$$\boxed{\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{m^2}{2}A_\mu A^\mu\,.} \tag{5.181}$$

The 'kinetic term' $F_{\mu\nu}F^{\mu\nu}$ can also be written as

$$\begin{aligned} F_{\mu\nu}F^{\mu\nu} &= F_{\mu\nu}(\partial^\nu A^\mu - \partial^\mu A^\nu) \\ &= F_{\mu\nu}\partial^\nu A^\mu - \underbrace{F_{\mu\nu}}_{-F_{\nu\mu}}\partial^\mu A^\nu \\ &= 2F_{\mu\nu}\partial^\nu A^\mu = 2(\partial_\nu A_\mu - \partial_\mu A_\nu)\partial^\nu A^\mu\,; \end{aligned} \tag{5.182}$$

namely,

$$\mathcal{L} = \frac{1}{2}\Big[(\partial_\mu A_\nu - \partial_\nu A_\mu)\partial^\nu A^\mu + m^2 A_\mu A^\mu\Big] \tag{5.183}$$

Let's apply the Euler-Lagrange equation to derive the equation of motion:

$$\frac{\partial\mathcal{L}}{\partial A_\alpha} = m^2 A^\alpha \tag{5.184}$$

and with some care,

$$\begin{aligned} \frac{\partial\mathcal{L}}{\partial(\partial_\beta A_\alpha)} &= \frac{1}{2}\frac{\partial}{\partial(\partial_\beta A_\alpha)}\Big[(\partial_\mu A_\nu - \partial_\nu A_\mu)\partial^\nu A^\mu\Big] \\ &= \frac{1}{2}\frac{\partial}{\partial(\partial_\beta A_\alpha)}\partial_\mu A_\nu \partial^\nu A^\mu - \frac{1}{2}\frac{\partial}{\partial(\partial_\beta A_\alpha)}\partial_\nu A_\mu \partial^\nu A^\mu \\ &= \partial^\alpha A^\beta - \partial^\beta A^\alpha \\ &= F^{\beta\alpha} = -F^{\alpha\beta}\,. \end{aligned} \tag{5.185}$$

Thus, we obtain

$$\frac{\partial \mathcal{L}}{\partial A_\alpha} = \partial_\beta \frac{\partial \mathcal{L}}{\partial(\partial_\beta A_\alpha)} \quad \rightarrow \quad \boxed{\partial_\beta F^{\alpha\beta} + m^2 A^\alpha = 0}\,, \tag{5.186}$$

which is the equation of motion for a massive spin-1 particle and called the *Proca equation*. Taking ∂_α of this and assuming that m is non-zero,

$$\underbrace{\partial_\alpha \partial_\beta F^{\alpha\beta}}_{\rightarrow\ 0 \ \text{since} \begin{cases} \partial_\alpha\partial_\beta = \partial_\beta\partial_\alpha \\ F^{\alpha\beta} = -F^{\beta\alpha} \end{cases}} + m^2 \partial_\alpha A^\alpha = 0 \quad \rightarrow \quad \partial_\alpha A^\alpha = 0 \quad (m \neq 0)\,, \tag{5.187}$$

which is nothing but the Lorentz condition (5.173). Writing out $F^{\alpha\beta}$ in (5.186) and using $\partial_\alpha A^\alpha = 0$,

$$\begin{aligned} 0 &= \partial_\beta F^{\alpha\beta} + m^2 A^\alpha \\ &= \partial_\beta(\partial^\beta A^\alpha - \partial^\alpha A^\beta) + m^2 A^\alpha \\ &= \partial^2 A^\alpha - \partial_\alpha \underbrace{\partial_\beta A^\beta}_{0} + m^2 A^\alpha \\ &\rightarrow \quad (\partial^2 + m^2) A^\alpha = 0\,. \end{aligned} \tag{5.188}$$

Thus, the Lagrangian density (5.181) leads to the Klein-Gordon equation for each component and one degree of freedom is removed in a way consistent with the Lorentz condition. The Proca equation can then be thought of as a 4-component Klein-Gordon equation with the Lorentz condition built into it.

The fields conjugate to A^α can be obtained by simply setting $\beta = 0$ in (5.185):

$$\pi^\alpha \equiv \frac{\partial \mathcal{L}}{\partial \dot{A}_\alpha} = F^{0\alpha}\,, \tag{5.189}$$

or

$$\pi^0 = 0\,, \qquad \pi^i = F^{0i}\,. \tag{5.190}$$

Thus, as promised, there is no conjugate field for A^0. In fact, A^0 can be derived from the rest from the fields in the Hamiltonian formalism: setting $\alpha = 0$ in the Proca equation (5.186) and using $F^{00} = 0$, we have

$$m^2 A^0 = -\partial_i \underbrace{F^{0i}}_{\pi^i} = -\vec{\nabla} \cdot \vec{\pi}\,, \tag{5.191}$$

where we note that $\vec{\nabla} \cdot \vec{\pi}$ is the difference of neighboring values of $\vec{\pi}$ and not considered to be a new independent field.

Momentum expansion and quantization

Now we will momentum-expand an arbitrary solution of the Proca equation $A^\mu(x)$, namely, a real 4-component field that simultaneously satisfies the Klein-Gordon equation and the Lorentz condition. Following the same procedure as in the case of the real spin-0 field (4.167), each component of $A^\mu(x)$, which satisfies the Klein-Gordon equation, can be uniquely expanded using the normal-mode functions $e_{\vec{p}}(x)$ and $e^*_{\vec{p}}(x)$. Thus, we have

$$A^\mu(x) = \sum_{\vec{p}} \left(A^\mu_{\vec{p}} e_{\vec{p}}(x) + A^{\mu *}_{\vec{p}} e^*_{\vec{p}}(x) \right) \qquad (\mu = 0, 1, 2, 3) \,. \tag{5.192}$$

At this point, $A^\mu_{\vec{p}}$ is a complex 4-vector. The Lorentz condition $\partial_\mu A^\mu(x) = 0$ then becomes

$$\partial_\mu A^\mu(x) = -i \sum_{\vec{p}} \left(p_\mu A^\mu_{\vec{p}} e_{\vec{p}}(x) - p_\mu A^{\mu *}_{\vec{p}} e^*_{\vec{p}}(x) \right) = 0 \,, \tag{5.193}$$

where $p^0 \equiv \sqrt{\vec{p}^{\,2} + m^2}$ as before. Applying $\int d^3x e^*_{\vec{p}}(x) i \overleftrightarrow{\partial}_0$ and using the orthonormality (4.175), the Lorentz condition translates to

$$p_\mu A^\mu_{\vec{p}} = 0 \quad (\text{for all } \vec{p}) \,. \tag{5.194}$$

The 4-component quantity $A^\mu_{\vec{p}}$ that satisfies this condition has three degrees of freedom, and we want to expand it using some three orthonormal 4-vectors. The situation is similar to the momentum expansion of the Dirac field where any complex 4-spinor was uniquely written in terms of the orthonormal set $(u_{\vec{p},\pm\vec{s}}, v_{-\vec{p},\pm\vec{s}})$. We will find the unique expansion of $A^\mu_{\vec{p}}$ as follows: We first boost p^μ and A^μ into the frame where p^μ is at rest, namely $p^\mu = (m, \vec{0}\,)$. In that frame, the condition $p_\mu A^\mu_{\vec{p}} = 0$ becomes

$$m A^0_{\vec{p}} = 0 \quad \to \quad A^\mu_{\vec{p}} = (0, \vec{A}_{\vec{p}}) \quad (\text{rest frame}) \,. \tag{5.195}$$

Then, the 3-vector $\vec{A}_{\vec{p}}$ can be expanded using three orthogonal unit vectors $\hat{\epsilon}_{\vec{p}1}$, $\hat{\epsilon}_{\vec{p}2}$, and $\hat{\epsilon}_{\vec{p}3}$. To define them uniquely, we take $\hat{\epsilon}_{\vec{p}3}$ in the $\vec{p}$ direction, $\hat{\epsilon}_{\vec{p}2}$ in the direction of $\vec{p} \times \hat{z}$, and then $\hat{\epsilon}_{\vec{p}1}$ is taken so that the three unit vectors $\hat{\epsilon}_{\vec{p}i}$ $(i = 1, 2, 3)$ form a right-handed coordinate system. In the following, the specific way to define the azimuthal orientations of $\hat{\epsilon}_{\vec{p}1}$ and $\hat{\epsilon}_{\vec{p}2}$ is not important as long as $\hat{\epsilon}_{\vec{p}3}$ is in the $\vec{p}$ direction. Thus, $\vec{A}_{\vec{p}}$ in the rest frame can be uniquely expanded as

$$\vec{A}_{\vec{p}} = \sum_{\lambda=1}^{3} a_{\vec{p}\lambda} \hat{\epsilon}_{\vec{p}\lambda} \quad (\text{rest frame}) \,, \tag{5.196}$$

or including the time component (which is zero),

$$A^\mu_{\vec{p}} = \sum_{\lambda=1}^{3} a_{\vec{p}\lambda} \epsilon^\mu_{\vec{p}\lambda} \,, \tag{5.197}$$

with

$$\epsilon^\mu_{\vec p\lambda} = (0, \hat\epsilon_{\vec p\lambda}) \quad (\lambda = 1, 2, 3\,,\ \text{rest frame})\,. \tag{5.198}$$

The original $A^\mu_{\vec p}$ is obtained by boosting (5.197) back to the original frame, which can be accomplished simply by boosting $\epsilon^\mu_{\vec p\lambda}$ in the expansion while keeping the same expansion coefficients $a_{\vec p\lambda}$. The boost is given by

$$\begin{aligned} \begin{pmatrix} \epsilon^{0\prime} \\ \epsilon'_\parallel \end{pmatrix} &= \begin{pmatrix} \gamma & \eta \\ \eta & \gamma \end{pmatrix} \begin{pmatrix} \epsilon^0 \\ \epsilon_\parallel \end{pmatrix} \\ \vec\epsilon\,'_\perp &= \vec\epsilon_\perp \end{aligned}\,, \quad \text{with} \quad \eta^\mu \equiv (\gamma, \vec\eta) \equiv \Big(\frac{p^0}{m}, \frac{\vec p}{m}\Big)\,. \tag{5.199}$$

The boosted 4-vectors in the original frame are then (dropping the primes)

$$\begin{aligned} \epsilon^\mu_{\vec p1} &= (0, \hat\epsilon_{\vec p1})\,, \\ \epsilon^\mu_{\vec p2} &= (0, \hat\epsilon_{\vec p2})\,, \\ \epsilon^\mu_{\vec p3} &= (\eta, \gamma\,\hat p)\,, \end{aligned} \tag{5.200}$$

forming a basis called the *linear basis*. These 4-vectors are sometimes called the polarization 4-vectors. The original $A^\mu_{\vec p}$ is thus uniquely expanded as (5.197) where $\epsilon^\mu_{\vec p\lambda}$ are taken to be the boosted polarization vectors given above. Then, the expansion of $A^\mu(x)$ (5.192) is now written as

$$A^\mu(x) = \sum_{\vec p\lambda} \Big(a_{\vec p\lambda} \epsilon^\mu_{\vec p\lambda}\, e_{\vec p}(x) + a^*_{\vec p\lambda} \epsilon^{\mu *}_{\vec p\lambda}\, e^*_{\vec p}(x) \Big) \tag{5.201}$$

or

$$\boxed{A^\mu(x) = \sum_{\vec p\lambda} \Big(a_{\vec p\lambda} h^\mu_{\vec p\lambda}(x) + a^\dagger_{\vec p\lambda} h^{\mu *}_{\vec p\lambda}(x) \Big)} \tag{5.202}$$

with

$$h^\mu_{\vec p\lambda}(x) \overset{\text{def}}{\equiv} \epsilon^\mu_{\vec p\lambda} e_{\vec p}(x)\,. \tag{5.203}$$

We have written $a^\dagger_{\vec p\lambda}$ instead of $a^*_{\vec p\lambda}$ anticipating the quantization of the field. Alternatively, one can define the *helicity basis* as

$$\begin{aligned} \epsilon^\mu_{\vec p+} &\overset{\text{def}}{\equiv} -\tfrac{1}{\sqrt2}(\epsilon^\mu_{\vec p1} + i\epsilon^\mu_{\vec p2}) \\ \epsilon^\mu_{\vec p0} &\overset{\text{def}}{\equiv} \epsilon^\mu_{\vec p3} \\ \epsilon^\mu_{\vec p-} &\overset{\text{def}}{\equiv} +\tfrac{1}{\sqrt2}(\epsilon^\mu_{\vec p1} - i\epsilon^\mu_{\vec p2}) \end{aligned} \tag{5.204}$$

In the rest frame of the particle, the space components are then (the time components are zero)

$$\begin{aligned} \vec\epsilon_{\vec p+} &= -\tfrac{1}{\sqrt2}(\hat\epsilon_{\vec p1} + i\hat\epsilon_{\vec p2}) \\ \vec\epsilon_{\vec p0} &= \hat\epsilon_{\vec p3} \\ \vec\epsilon_{\vec p-} &= +\tfrac{1}{\sqrt2}(\hat\epsilon_{\vec p1} - i\hat\epsilon_{\vec p2}) \end{aligned} \qquad (\text{rest frame} : \vec p = 0)\,. \tag{5.205}$$

We will now show that these polarization vectors correspond to the eigenstates $|j,m\rangle = |1,1\rangle$, $|1,0\rangle$, $|1,-1\rangle$, where the spin axis is taken as $\hat{\epsilon}_{\vec{p}3}$ (namely, J_3 represents the helicity). Using the explicit expressions for $\vec{J}$ (5.169), we see that the eigenstate $|1,0\rangle$ is given by $(0,0,1)$:

$$J_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = i \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = 0 \quad \to \quad |1,0\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} . \tag{5.206}$$

The states $|1,1\rangle$ and $|1,-1\rangle$ can be constructed from $|1,0\rangle$ by applying the raising and lowering operators:

$$J_\pm \equiv J_1 \pm iJ_2 = \begin{pmatrix} 0 & 0 & \mp 1 \\ 0 & 0 & -i \\ \pm 1 & i & 0 \end{pmatrix} , \tag{5.207}$$

whose action on $|j,m\rangle$ is given in general by

$$\sqrt{j(j+1) - m(m\pm 1)}|j, m\pm 1\rangle = J_\pm |j,m\rangle .$$

For $|j,m\rangle = |1,0\rangle$, we have $j(j+1) - m(m\pm 1) = 2$, and thus

$$\sqrt{2}|1,\pm 1\rangle = J_\pm |1,0\rangle = J_\pm \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \mp 1 \\ -i \\ 0 \end{pmatrix} \quad \to \quad |1,\pm 1\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} \mp 1 \\ -i \\ 0 \end{pmatrix} . \tag{5.208}$$

Namely, $|1,1\rangle$, $|1,0\rangle$, and $|1,-1\rangle$ are represented in the three-dimensional space by

$$|1,1\rangle = -\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix} , \quad |1,0\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} , \quad |1,-1\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix} . \tag{5.209}$$

These are nothing but the polarization vectors $\hat{\epsilon}_{\vec{p}+}$, $\hat{\epsilon}_{\vec{p}0}$, and $\hat{\epsilon}_{\vec{p}-}$ given in (5.205) where the coordinate axes are $\hat{\epsilon}_{\vec{p}i}$ $(i = 1,2,3)$.

In the helicity basis, the corresponding expansion coefficients are given by

$$\begin{aligned} a_{\vec{p}+} &= -\tfrac{1}{\sqrt{2}}(a_{\vec{p}1} - i a_{\vec{p}2}) \\ a_{\vec{p}0} &= a_{\vec{p}3} \\ a_{\vec{p}-} &= +\tfrac{1}{\sqrt{2}}(a_{\vec{p}1} + i a_{\vec{p}2}) \end{aligned} \tag{5.210}$$

Then in the momentum expansion (5.201), the sum over the linear basis becomes the same as the sum over the helicity basis:

$$a_{\vec{p}+}\epsilon^\mu_{\vec{p}+} + a_{\vec{p}-}\epsilon^\mu_{\vec{p}-} = a_{\vec{p}1}\epsilon^\mu_{\vec{p}1} + a_{\vec{p}2}\epsilon^\mu_{\vec{p}2} , \quad a_{\vec{p}0}\epsilon^\mu_{\vec{p}0} = a_{\vec{p}3}\epsilon^\mu_{\vec{p}3} ,$$

$$\rightarrow \quad \sum_{\lambda=1,2,3} a_{\vec{p}\lambda}\epsilon^{\mu}_{\vec{p}\lambda} = \sum_{\lambda=+,0,-} a_{\vec{p}\lambda}\epsilon^{\mu}_{\vec{p}\lambda}\,. \tag{5.211}$$

Thus, the momentum expansion (5.202) is valid for $\lambda = +, 0, -$ (helicity basis) as well as for $\lambda = 1, 2, 3$ (linear basis). Also, the relations

$$\boxed{\epsilon_{\vec{p}\lambda}\cdot\epsilon^{*}_{\vec{p}\lambda'} = -\delta_{\lambda\lambda'}\,, \quad p\cdot\epsilon_{\vec{p}\lambda} = 0\,,} \tag{5.212}$$

hold for $\lambda = 1, 2, 3$ and $+, 0, -$, which can be trivially proven for the linear basis in the rest frame which can then be extended to the helicity basis using (5.204).

In dealing with massive spin-1 particles we often encounter a polarization sum of the form

$$\sum_{\lambda} \epsilon^{\mu}_{\vec{p}\lambda}\epsilon^{\nu *}_{\vec{p}\lambda}\,, \tag{5.213}$$

where the polarization vectors refer to a given particle (not different particles). This is the spin-1 equivalent of $\sum_{\vec{s}} u_{\vec{p},\vec{s}}\bar{u}_{\vec{p},\vec{s}}$ and $\sum_{\vec{s}} v_{\vec{p},\vec{s}}\bar{v}_{\vec{p},\vec{s}}$. To evaluate this, we first note that it is a Lorentz tensor; thus, we can evaluate it in the rest frame ($\vec{p} = 0$) and express it in a Lorentz-covariant form, then it will be valid in any frame. In the rest frame and using the linear basis, we have

$$\begin{array}{rcl} \mu & : & 0,1,2,3 \\ \eta^{\mu}_{\vec{0}} & = & (1,0,0,0) \\ \epsilon^{\mu}_{\vec{0}1} & = & (0,1,0,0) \\ \epsilon^{\mu}_{\vec{0}2} & = & (0,0,1,0) \\ \epsilon^{\mu}_{\vec{0}3} & = & (0,0,0,1)\,, \end{array} \tag{5.214}$$

where $\eta_{\vec{0}}$ is $\eta^{\mu} \equiv p^{\mu}/m$ evaluated in the rest frame. If we regard each column as a 4-vector and take the inner product of μ-th column and ν-th column, we see that the following expression holds numerically:

$$\eta^{\mu}_{\vec{0}}\eta^{\nu}_{\vec{0}} - \sum_{\lambda=1,2,3} \epsilon^{\mu}_{\vec{0}\lambda}\epsilon^{\nu}_{\vec{0}\lambda} = g^{\mu\nu} \quad \rightarrow \quad \sum_{\lambda=1,2,3} \epsilon^{\mu}_{\vec{0}\lambda}\epsilon^{\nu}_{\vec{0}\lambda} = -g^{\mu\nu} + \eta^{\mu}_{\vec{0}}\eta^{\nu}_{\vec{0}}\,. \tag{5.215}$$

Now we boost the above equation on the right, or more precisely, multiply both sides of the equality by $\Lambda^{\alpha}{}_{\mu}\Lambda^{\beta}{}_{\nu}$ and sum over μ and ν, where Λ is the boost that takes the rest mass m to p^{μ}:

$$\sum_{\lambda=1,2,3} \underbrace{\Lambda^{\alpha}{}_{\mu}\epsilon^{\mu}_{\vec{0}\lambda}}_{\epsilon^{\alpha}_{\vec{p}\lambda}} \underbrace{\Lambda^{\beta}{}_{\nu}\epsilon^{\nu}_{\vec{0}\lambda}}_{\epsilon^{\beta}_{\vec{p}\lambda}} = -\underbrace{\Lambda^{\alpha}{}_{\mu}\Lambda^{\beta}{}_{\nu}g^{\mu\nu}}_{g^{\alpha\beta}} + \underbrace{\Lambda^{\alpha}{}_{\mu}\eta^{\mu}_{\vec{0}}}_{\eta^{\alpha}} \underbrace{\Lambda^{\beta}{}_{\nu}\eta^{\nu}_{\vec{0}}}_{\eta^{\beta}}\,; \tag{5.216}$$

namely,

$$\sum_{\lambda=1,2,3} \epsilon^{\mu}_{\vec{p}\lambda}\epsilon^{\nu}_{\vec{p}\lambda} = -g^{\mu\nu} + \eta^{\mu}\eta^{\nu}\,. \tag{5.217}$$

Using the definition of the helicity basis (5.204), we can show that the helicity basis version is related to the linear basis version by

$$\sum_{\lambda=+,0,-} \epsilon^{\mu}_{\vec{p}\lambda}\epsilon^{\nu *}_{\vec{p}\lambda} = \sum_{\lambda=1,2,3} \epsilon^{\mu}_{\vec{p}\lambda}\epsilon^{\nu}_{\vec{p}\lambda} \,. \tag{5.218}$$

Then, the relation

$$\boxed{\sum_{\lambda} \epsilon^{\mu}_{\vec{p}\lambda}\epsilon^{\nu *}_{\vec{p}\lambda} = -g^{\mu\nu} + \frac{p^{\mu}p^{\nu}}{m^2} \qquad (\lambda = 1,2,3 \;\text{ or }\; +,0,-)} \tag{5.219}$$

works for both the linear basis and the helicity basis.

Now, we quantize the field by regarding the expansion coefficients as and introducing commutators among them. Since the spin is an integer, we have to use commutators instead of anticommutators.

$$\boxed{\begin{array}{rcl} [a_{\vec{p}\lambda}, a^{\dagger}_{\vec{p}'\lambda'}] &=& \delta_{\vec{p},\vec{p}'}\delta_{\lambda,\lambda'} \,, \\ [a_{\vec{p}\lambda}, a_{\vec{p}'\lambda'}] &=& [a^{\dagger}_{\vec{p}\lambda}, a^{\dagger}_{\vec{p}'\lambda'}] = 0 \,, \end{array}} \tag{5.220}$$

where one can use either linear or helicity bases, and both are consistent. With the help of the polarization sum formula above, this set of commutators leads to equal-time commutators among fields given by

$$\begin{array}{rcl} [A_i(t,\vec{x}), \pi^j(t,\vec{x}\,')] &=& i g_i{}^j \delta^3(\vec{x}-\vec{x}\,') \,, \\ [A_i(t,\vec{x}), A^j(t,\vec{x}\,')] &=& [\pi_i(t,\vec{x}), \pi^j(t,\vec{x}\,')] = 0 \,. \end{array} \tag{5.221}$$

Exercise 5.4 *Quantization of massive spin-1 field.*
(*a*) *Use the commutation relations of creation and annihilation operators (5.220) and the momentum expansion (5.202), where $\lambda = +,0,-$ or $1,2,3$, to derive the equal-time commutation relations (5.221).*
(*b*) *Show that the commutation relation between $A^{\mu}(x)$ and $A^{\nu}(x')$ at two space-time points x and x' is given by*

$$[A^{\mu}(x), A^{\nu}(x')] = \Big(-g^{\mu\nu} - \frac{\partial^{\mu}_{x}\partial^{\nu}_{x}}{m^2}\Big) i\Delta(x-x') \,. \tag{5.222}$$

Is the microscopic causality satisfied?
(*hint: The conjugate field can be written as*

$$\pi^i = -i \sum_{\vec{p},\lambda} (a_{\vec{p},\lambda} F^{i}_{\vec{p},\lambda} - a^{\dagger}_{\vec{p},\lambda} F^{i*}_{\vec{p},\lambda}) \quad \textit{with} \quad F^{i}_{\vec{p},\lambda} \equiv p^i h^0_{\vec{p},\lambda} - p^0 h^i_{\vec{p},\lambda} \,. \tag{5.223}$$

You will need the polarization sum formula $\sum_{\lambda} \epsilon^{\mu}_{\vec{p}\lambda}\epsilon^{\nu *}_{\vec{p}\lambda} = -g^{\mu\nu} + p^{\mu}p^{\nu}/m^2$.)

Charged spin-1 field

Just as we combined two real scalar fields of same mass to form a single complex field that represents a 'charged' scalar particle, we can combine two real vector fields to form a single complex vector field that describes a charged spin-1 particle. Such a field can be used when particle and antiparticle are distinct such as $W^{\pm}$ bosons. Thus, we combine two real vector fields A^1_μ and A^2_μ as

$$A_\mu(x) = \frac{1}{\sqrt{2}}\Big(A^1_\mu(x) + iA^2_\mu(x)\Big)\,. \tag{5.224}$$

The Lagrangian density of the whole is simply the sum of the Lagrangian density of A^1_μ and that of A^2_μ:

$$\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2 = -\frac{1}{4}F^i_{\mu\nu}F^{i\mu\nu} + \frac{m^2}{2}A^i_\mu A^{i\mu} \tag{5.225}$$

where the sum over i is implied and

$$F^i_{\mu\nu} \stackrel{\text{def}}{\equiv} \partial_\nu A^i_\mu - \partial_\mu A^i_\nu \quad (i=1,2) \quad (\text{real})\,. \tag{5.226}$$

Defining the antisymmetric tensor $F_{\mu\nu}$ for the complex field A_μ in the same way,

$$F_{\mu\nu} \stackrel{\text{def}}{\equiv} \partial_\nu A_\mu - \partial_\mu A_\nu \quad (\text{complex})\,, \tag{5.227}$$

we have

$$F_{\mu\nu} = \frac{1}{\sqrt{2}}\Big[\partial_\nu(A^1_\mu + iA^2_\mu) - \partial_\mu(A^1_\nu + iA^2_\nu)\Big] = \frac{1}{\sqrt{2}}(F^1_{\mu\nu} + iF^2_{\mu\nu})\,; \tag{5.228}$$

then the kinetic term can be written as

$$F^*_{\mu\nu}F^{\mu\nu} = \frac{1}{2}(F^1_{\mu\nu} - iF^2_{\mu\nu})(F^{1\mu\nu} + iF^{2\mu\nu}) = \frac{1}{2}F^i_{\mu\nu}F^{i\mu\nu}\,. \tag{5.229}$$

Similarly, we have

$$A^*_\mu A^\mu = \frac{1}{2}(A^1_\mu - iA^2_\mu)(A^{1\mu} + iA^{2\mu}) = \frac{1}{2}A^i_\mu A^{i\mu}\,. \tag{5.230}$$

Thus, the total Lagrangian density can be written as

$$\mathcal{L} = -\frac{1}{2}F^*_{\mu\nu}F^{\mu\nu} + m^2 A^*_\mu A^\mu\,. \tag{5.231}$$

The momentum expansion of the complex (namely, non-hermitian) vector field proceeds similarly to the charged scalar field case: we define

$$\begin{aligned} a_{\vec{p}\lambda} &\stackrel{\text{def}}{\equiv} \frac{1}{\sqrt{2}}(a_{1\vec{p}\lambda} + ia_{2\vec{p}\lambda}) \\ b_{\vec{p}\lambda} &\stackrel{\text{def}}{\equiv} \frac{1}{\sqrt{2}}(a_{1\vec{p}\lambda} - ia_{2\vec{p}\lambda}) \end{aligned} \tag{5.232}$$

where $a_{i\vec{p}\lambda}$ are the annihilation operators for the vector field $A^i_\mu(x)$. Then, using the expansion (5.202), the non-hermitian field A^μ is written as

$$\boxed{A^\mu(x) = \sum_{\vec{p}\lambda} \Big(a_{\vec{p}\lambda} h^\mu_{\vec{p}\lambda}(x) + b^\dagger_{\vec{p}\lambda} h^{\mu *}_{\vec{p}\lambda}(x) \Big)}\,, \tag{5.233}$$

where the normal-mode functions $h^\mu_{\vec{p}\lambda}(x)$ are the same as before. Using the commutation relations (5.220) for each of the two real fields together with the assumption that operators belonging to different fields commute, we obtain

$$\boxed{\begin{array}{c} [a_{\vec{p}\lambda}, a^\dagger_{\vec{p}'\lambda'}] = [b_{\vec{p}\lambda}, b^\dagger_{\vec{p}'\lambda'}] = \delta_{\vec{p},\vec{p}'}\delta_{\lambda,\lambda'}\,, \\ \text{all others } = 0\,. \end{array}} \tag{5.234}$$

As in the case of a charged scalar field, $a^\dagger_{\vec{p}\lambda}$ is regarded as a creation operator of a particle and $b^\dagger_{\vec{p}\lambda}$ as that of its antiparticle. This interpretation is justified by the conserved quantity Q corresponding to the invariance of the Lagrangian density under the phase transformation

$$A'_\mu(x) = e^{i\theta} A_\mu(x)\,. \tag{5.235}$$

The same procedure as before leads to

$$Q = \sum_{\vec{p}\lambda} (a^\dagger_{\vec{p}\lambda} a_{\vec{p}\lambda} - b^\dagger_{\vec{p}\lambda} b_{\vec{p}\lambda}) \tag{5.236}$$

which shows that an a-particle carries charge $+1$ and a b-particle carries charge -1 regardless of momentum and spin.

Weak decays - W-fermion coupling

Weak decays are mostly caused by couplings between the massive charged spin-1 particle $W^\pm$ and fermion currents. Decays such as $W^+ \to e^+\nu_e$, $\pi^+ \to \mu^+\nu_\mu$, and $\mu^- \to \nu_\mu e^- \bar{\nu}_e$ are some examples. Often, W-fermion couplings are hidden inside effective couplings. For example, even though π^+ is a spin-0 particle, quarks (which are fermions) inside couple to W^+ which then creates the $\mu^+\nu_\mu$ pair. In $\mu^- \to \nu_\mu e^- \bar{\nu}_e$, the current $\mu^- \to \nu_\mu$ couples to W^- which then creates the $e^-\bar{\nu}_e$ pair.

The W^+ particle can be described by the charged spin-1 field introduced in the previous section. The $W - e\nu_e$ coupling is a $V - A$ coupling given by

$$\mathcal{L}_{\rm int} = \frac{g}{2\sqrt{2}} \Big(\bar{\nu}_e \gamma_\mu (1 - \gamma_5) e \Big) W^\mu + h.c. \tag{5.237}$$

where $W^\mu(x)$ is the charged vector field for $W^\pm$ ($W^+ =$ particle, $W^- =$ antiparticle) and $\nu_e(x)$ and $e(x)$ are the short hands for $\psi_{\nu_e}(x)$ and $\psi_e(x)$, respectively. The constant g is the universal coupling constant $g \sim 0.65$.

In the standard model, leptons appear in three 'generations':

$$\begin{matrix} Q=0 \\ Q=-1 \end{matrix} \quad \begin{pmatrix} \nu_e \\ e^- \end{pmatrix} \quad \begin{pmatrix} \nu_\mu \\ \mu^- \end{pmatrix} \quad \begin{pmatrix} \nu_\tau \\ \tau^- \end{pmatrix}, \tag{5.238}$$

where Q is the electrical charge and all particles shown are fermions as opposed to antifermions. They have exactly the same types of interactions, including weak interactions, with only difference being the charged lepton masses

$$m_e = 0.000511\mathrm{GeV}\,, \quad m_\mu = 0.106\mathrm{GeV}\,, \quad m_\tau = 1.78\mathrm{GeV}\,. \tag{5.239}$$

Unless otherwise stated, we assume that all neutrinos are massless. Thus, W-lepton coupling is written as

$$\boxed{\mathcal{L}_{\rm int} = \frac{g}{2\sqrt{2}}\Big(\bar\nu_i\gamma_\mu(1-\gamma_5)\ell_i\Big)W^\mu + h.c. \quad (i=1,2,3)}\,, \tag{5.240}$$

with

$$(\nu_1,\nu_2,\nu_3) \equiv (\nu_e,\nu_\mu,\nu_\tau)\,, \qquad (\ell_1,\ell_2,\ell_3) \equiv (e,\mu,\tau)\,, \tag{5.241}$$

and the sum over $i=1,2,3$ is implied. Quarks also come in three generations,

$$\begin{matrix} Q=+2/3 \\ Q=-1/3 \end{matrix} \quad \begin{pmatrix} u \\ d \end{pmatrix} \quad \begin{pmatrix} c \\ s \end{pmatrix} \quad \begin{pmatrix} t \\ b \end{pmatrix}, \tag{5.242}$$

and couple to W in a similar way:

$$\boxed{\mathcal{L}_{\rm int} = \frac{g}{2\sqrt{2}}V_{ij}\Big(\bar U_i\gamma_\mu(1-\gamma_5)D_j\Big)W^\mu + h.c. \quad (i=1,2,3)}\,, \tag{5.243}$$

where

$$(U_1,U_2,U_3) \equiv (u,c,t)\,, \qquad (D_1,D_2,D_3) \equiv (d,s,b)\,, \tag{5.244}$$

and sum over $i,j=1,2,3$ is implied. The set of constants V_{ij} is called the Cabibbo-Kobayashi-Masukawa matrix (the CKM matrix) and the element V_{ij} specifies the strength of coupling between the U_i-D_j current and W in units of the W-$e\nu_e$ coupling. It turns out that V is unitary (the standard model requires it theoretically, but it is also consistent with experiment) and that the diagonal elements V_{ii} are nearly unity while cross-generational couplings are suppressed but finite. Note that in the W-lepton couplings there is no cross-generational coupling. Approximately, the sizes of $|V_{ij}|$ are given by

$$|V_{ij}| \sim \begin{pmatrix} 1 & \lambda & \lambda^3 \\ \lambda & 1 & \lambda^2 \\ \lambda^3 & \lambda^2 & 1 \end{pmatrix} \quad (\lambda \sim 0.22)\,. \tag{5.245}$$

Also, it is believed that there are relative complex phases between V_{ij} which result in violation of CP symmetry. For example, the W-tb coupling is given by

$$\mathcal{L} = \frac{g}{2\sqrt{2}} \underbrace{V_{tb}}_{\sim 1} \left(\bar{t}\gamma_\mu(1-\gamma_5)b\right) W^\mu + h.c. \tag{5.246}$$

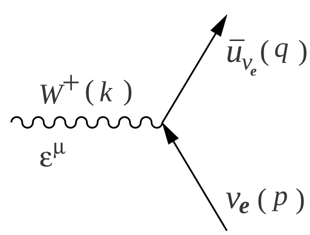
Actually, quarks come in three colors, and the same interaction above repeats for each color. That is, the current $\bar{U}_i\gamma_\mu(1-\gamma_5)D_j$ is understood to be

$$\bar{U}_i\gamma_\mu(1-\gamma_5)D_j \equiv \sum_a \bar{U}_i^a \gamma_\mu(1-\gamma_5)D_j^a \tag{5.247}$$

where the sum is over a = red, blue, green. The W-quark coupling does not change color, namely there is no coupling with color-changing currents.

W decays

We will first calculate the decay $W^+ \to e^+\nu_e$. Other decays such as $W^+ \to \mu^+\nu_\mu$ are essentially the same. The interaction responsible is given by (5.237) or using the projection operator P_L,



$$\mathcal{L}_{\rm int} = \frac{g}{\sqrt{2}}(\bar{\nu}_e\gamma_\mu P_L e)W^\mu\,. \tag{5.248}$$

What is the Lorentz-invariant matrix element? The situation is nearly identical to that of $H \to t\bar{t}$. This time we have

$$\langle f|\mathcal{L}_{\rm int}|i\rangle = \langle 0|b_e a_{\nu_e} \frac{g}{\sqrt{2}}(\bar{\nu}_e\gamma_\mu P_L e)W^\mu a_W^\dagger|0\rangle \tag{5.249}$$

where the subscripts e, ν_e, and W for the creation or annihilation operators indicate that they are for the corresponding particles of the initial and final states. The rule for the final state fermion or antifermion stays the same; namely, ν_e will pick up $\bar{u}_{\nu_e}$ and e^+ will pick up v_e. When the vector field W^μ is expanded as

$$W^\mu = \sum_{\vec{p}\lambda}\left(a_{\vec{p}\lambda}h^\mu_{\vec{p}\lambda} + b^\dagger_{\vec{p}\lambda}h^{\mu *}_{\vec{p}\lambda}\right), \tag{5.250}$$

the annihilation operator $a_{\vec{k}\lambda}$ that matches $a_W^\dagger$ ($\equiv a^\dagger_{\vec{k}\lambda}$) of the initial state comes with the factor $h^\mu_{\vec{k}\lambda}(x) = \epsilon^\mu_{\vec{k}\lambda}e_{\vec{k}}(x)$. The $e_{\vec{k}}(x)$ factor will become part of the delta function for energy-momentum conservation and the normalization in the definition of $\mathcal{M}$ and $\epsilon^\mu_{\vec{k}\lambda}$ will be included in $\mathcal{M}$:

$$S_{fi} = i\int d^4x\langle f|\mathcal{L}_{\rm int}|i\rangle = \frac{(2\pi)^4\delta^4(p+q-k)}{\sqrt{(2p^0V)(2q^0V)(2k^0V)}}\,\mathcal{M} \tag{5.251}$$

with

$$\mathcal{M} = i\frac{g}{\sqrt{2}}\,(\bar{u}_{\nu_e}\gamma_\mu P_L v_e)\epsilon^\mu \tag{5.252}$$

where $\epsilon^\mu \equiv \epsilon^\mu_{\vec{k}\lambda}$. If the initial state is W^-, then the matching annihilation operator $b_{\vec{k}\lambda}$ is in $W^{\mu\dagger}$ which appears in the hermitian conjugate term, and the normal-mode function associated is again $h^\mu_{\vec{k}\lambda}(x)$. We note that regardless of particle or antiparticle, an annihilation operator is associated with $h^\mu_{\vec{p}\lambda}$ and a creation operator with $h^{\mu*}_{\vec{p}\lambda}$. Thus, the rule is to include in $\mathcal{M}$ the factor ϵ^μ for a spin-1 particle in the initial state, and the factor $\epsilon^{\mu*}$ for the final state regardless of particle or antiparticle:

	initial state	final state
spin-1	$\epsilon^\mu_{\vec{p}\lambda}$	$\epsilon^{\mu*}_{\vec{p}\lambda}$

. (5.253)

(for particle or antiparticle)

Let's proceed to calculate the decay rate of $W^+ \to e^+\nu_e$. Since $e^+\nu_e$ is created by a $V-A$ current, e^+ is right-handed (to the extent that m_e is small) and ν_e is left-handed. We will, however, sum over the spins of e^+ and ν_e to take advantage of the trace techniques. We assume that the initial state W^+ is unpolarized, and thus take average over the three possible helicity states or equivalently the three linear polarizations. Thus, we will evaluate the sum of $|\mathcal{M}|^2$ over all spins and divide by three to obtain the spin-averaged (unpolarized) $|\mathcal{M}|^2$: with $m_W \equiv m$ and $m_e = m_{\nu_e} = 0$,

$$\begin{aligned}
\overline{|\mathcal{M}|^2} &= \frac{1}{3}\sum_{\rm spins}|\mathcal{M}|^2 \\
&= \frac{g^2}{3\cdot 2}\sum_{\rm spins}(\bar{u}_{\nu_e}\gamma_\mu P_L v_e\,\epsilon^\mu)^*(\bar{u}_{\nu_e}\gamma_\nu P_L v_e\,\epsilon^\nu) \\
&= \frac{g^2}{6}\sum_{\rm spins}\bar{v}_e P_R\gamma_\mu u_{\nu_e}\bar{u}_{\nu_e}\gamma_\nu P_L v_e\underbrace{\sum_{\rm spins}\epsilon^{\mu*}\epsilon^\nu}_{-g^{\mu\nu}+\frac{k^\mu k^\nu}{m^2}} \\
&= \frac{g^2}{6}\Big[-\sum_{\rm spins}\bar{v}_e P_R\gamma_\mu u_{\nu_e}\bar{u}_{\nu_e}\gamma^\mu P_L v_e \\
&\qquad +\frac{1}{m^2}\sum_{\rm spins}\bar{v}_e P_R\not{k}u_{\nu_e}\bar{u}_{\nu_e}\not{k}P_L v_e\Big].
\end{aligned} \tag{5.254}$$

Now, the second term vanishes as follows: the Dirac equations in momentum space for u_{ν_e} and v_e are (with $m_e = m_{\nu_e} = 0$)

$$\not{p} v_e = 0 \,, \quad \not{q} u_{\nu_e} = 0 \,; \tag{5.255}$$

thus, using $k = p + q$, we have

$$\bar{v}_e P_R \not{k} u_{\nu_e} = \bar{v}_e P_R (\not{p} + \underbrace{\not{q}) u_{\nu_e}}_{0} = \bar{v}_e \underbrace{P_R \not{p}}_{\not{p} P_L} u_{\nu_e} = \underbrace{\overline{\not{p} v_e}}_{0} P_L u_{\nu_e} = 0 \,. \tag{5.256}$$

Incidentally, a similar calculation shows that

$$[\bar{w}_{\vec{p},\vec{s}} \gamma_\mu (a + b\gamma_5) w'_{\vec{p}',\vec{s}'}] k^\mu = 0 \quad (w, w' = u \text{ or } v, \text{ massless}) \tag{5.257}$$

where a and b are arbitrary constants and k is any linear combination of p and p'. Thus, discarding the second term in (5.254) and executing the spin sums, we have

$$\begin{aligned} \overline{|\mathcal{M}|^2} \;&=\; -\frac{g^2}{6} \sum_{\text{spins}} \bar{v}_e P_R \underbrace{\gamma_\mu \underbrace{u_{\nu_e} \bar{u}_{\nu_e}}_{\to \not{q}} \gamma^\mu}_{-2\not{q}} P_L v_e = \frac{g^2}{3} \mathrm{Tr} \not{p} \underbrace{\underbrace{P_R \not{q}}_{\not{q} P_L} P_L}_{\not{q} P_L} \\ &=\; \frac{g^2}{6} \underbrace{\mathrm{Tr} \not{p} \not{q} (1}_{4p\cdot q} - \cancel{\gamma_5}) = \frac{2g^2}{3} p \cdot q \,, \end{aligned} \tag{5.258}$$

where we have used $\gamma_\mu \not{a} \gamma^\mu = -2\not{a}$, $P_L^2 = P_L$, $\mathrm{Tr} \not{a} \not{b} \gamma_5 = 0$, and $\mathrm{Tr} \not{a} \not{b} = 4a \cdot b$.

Squaring $k = p + q$ and using $m_e = m_\nu = 0$, we get $m^2 = 2p \cdot q$:

$$p \cdot q = \frac{m^2}{2} \,. \tag{5.259}$$

Using the 2-body decay rate formula $\Gamma = (|\vec{p}|/8\pi m^2) \overline{|\mathcal{M}|^2}$, we obtain,

$$\Gamma(W^+ \to e^+ \nu_e) = \frac{g^2}{48\pi} m \,. \tag{5.260}$$

With $g = 0.65$ and $m = 80$ GeV, we obtain $\Gamma(W^+ \to e^+ \nu_e) = 0.224$ GeV. The total decay rate of W is obtained by summing up the partial decay rates over all possible final states, which are

$$\begin{array}{rccccccccc} \text{mode}: & e^+\nu_e & \mu^+\nu_\mu & \tau^+\nu_\tau & u\bar{d} & u\bar{s} & u\bar{b} & c\bar{d} & c\bar{s} & c\bar{b} \\ \Gamma: & 1 & 1 & 1 & |V_{ud}|^2 & |V_{us}|^2 & |V_{ub}|^2 & |V_{cd}|^2 & |V_{cs}|^2 & |V_{cb}|^2 \end{array} \tag{5.261}$$

where the decay rates are given in the unit of $\Gamma(W^+ \to e^+\nu_e)$. The W boson can also couple to $t\bar{d}$, $t\bar{s}$, and $t\bar{b}$; the mass of t (~ 175 GeV), however, is heavier than that of W, and thus such decays are prohibited. We have assumed that the fermion masses are small, which is a good assumption since the heaviest in the list is $m_b \sim 5$ GeV which is still much smaller than $m_W \sim 80$ GeV. Recalling that quarks come in three colors, and using the unitarity of the CKM matrix

$$\sum_i V_{ij}^* V_{ik} = \delta_{jk}\,, \tag{5.262}$$

the total decay rate is

$$\begin{aligned}\Gamma_{\rm tot} &= \Gamma(W^+ \to e^+\nu_e)\Big(\underset{\substack{\uparrow \\ e,\mu,\tau}}{3} + 3(\underbrace{|V_{ud}|^2 + |V_{us}|^2 + |V_{ub}|^2}_{1} + \underbrace{|V_{cd}|^2 + |V_{cs}|^2 + |V_{cb}|^2}_{1}\Big) \\ &= 2.02\ \text{GeV}\,. \end{aligned} \tag{5.263}$$

The experimental value is $\Gamma_{\rm tot} = 2.08 \pm 0.07$ GeV. Since the value of g is determined from elsewhere - as we will determine from the decay $\mu^- \to \nu_\mu e^- \bar{\nu}_e$ later - the agreement is quite remarkable. This also supports that quarks indeed come in three colors.

5.6 $\nu_\mu e^- \to \mu^- \nu_e$ scattering

Next, we will consider another W-mediated interaction $\nu_\mu e^- \to \mu^-\nu_e$. The main purposes are to introduce the *scattering cross section* and the *W-propagator*. Let's start from the first.

Scattering cross section

Consider a general scattering interaction $a + b \to 1 + 2 + \ldots + n$, where a is the projectile particle, b is the target particle, and $1, 2, \ldots, n$ are the final-state particles. The concept of scattering cross section is just about the simplest way to define the 'likelihood' that a projectile interacts with a target. When single projectile particle is traveling with velocity v in a uniform target medium where the density of the target is n particles per unit volume, the probability to interact within a time window T is proportional to the time duration T, to the velocity v, and to the density of the target n (assuming that the probability is much less than unity). When these 'trivial factors' are removed, then what is left is the scattering cross section which happens to have the dimension of area.

As a simple example, consider the classical case where each target is a sphere of radius r which is at rest in the laboratory frame, and the radius of the projectile is small enough to be neglected. The probability that the projectile will hit a target in

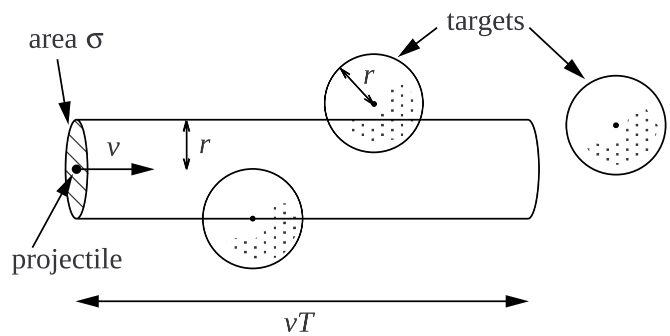


Figure 5.3: When a target center is located within the cylinder of cross section $\sigma = \pi r^2$ and length vT, the point projectile will hit the target.

time T is the probability that a center of target is located inside the cylinder of cross section $\sigma = \pi r^2$ and length vT (Figure 5.3). If the probability is small enough, we can neglect the chance of finding multiple targets in the cylinder. Then, the probability, or the expected number of collisions N^0, is given by

$$N^0 = n \times (\text{volume of the cylinder}) = n\,\sigma v T\,. \tag{5.264}$$

This formula is valid even when the target medium is also moving as long as the velocity of the projectile ($\vec{v}_a$) and that of target ($\vec{v}_b$) are parallel to each other. In such case, n is still measured in the laboratory frame, and the velocity v is understood to be the relative velocity of projectile and target measured in the laboratory frame:

$$v \stackrel{\text{def}}{\equiv} |\vec{v}_a - \vec{v}_b|\,. \tag{5.265}$$

Now, suppose that there are ρ projectiles in unit volume measured in the laboratory frame. Then, there are ρV projectiles in a volume V on average, and for each projectile, the probability to interact in time T is $N^0 = n\,\sigma v T$. Thus, the probability N to see an interaction in the volume V in the time duration T is

$$N = \rho V N^0 = \rho V n\,\sigma v T\,. \tag{5.266}$$

For general scattering $a + b \to 1 + 2 + \ldots + n$ where the target does not have a classical target area, this relation defines the cross section σ. The cross section $d\sigma$ for the final-state particles to scatter into the momentum ranges $\vec{p}_f \in d^3p_f$ $(f = 1, \ldots, n)$ is then defined by

$$dN = \rho V n\, d\sigma v T\,, \tag{5.267}$$

where dN is the probability to find the final state in the specified momentum ranges. On the other hand, the same quantity dN is given by summing $|S_{fi}|^2$ over the corresponding final states. Since the initial state $|i\rangle = a_a^\dagger a_b^\dagger |0\rangle$ corresponds to having one projectile partile and one target particle in the entire volume V, we have

$$\rho = \frac{1}{V} : \text{ projectile density}, \quad n = \frac{1}{V} : \text{ target density}. \tag{5.268}$$

Thus,

$$dN = \sum_{\vec{p}_f \in d^3p_f} |S_{fi}|^2 = \rho V n \, d\sigma \, vT = d\sigma \, \frac{vT}{V}. \tag{5.269}$$

Using the definition of $\mathcal{M}$ (5.107)

$$S_{fi} \equiv \frac{(2\pi)^4 \delta^4(p_a + p_b - \sum_f p_f)}{\sqrt{(2p_a^0 V)(2p_b^0 V) \prod_f (2p_f^0 V)}} \mathcal{M}, \tag{5.270}$$

and the identity (5.87), we have

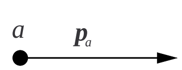
$$\begin{aligned} d\sigma &= \frac{V}{vT} \sum_{\vec{p}_f \in d^3p_f} |S_{fi}|^2 \\ &= \frac{V}{vT} \frac{\overbrace{[(2\pi)^4 \delta^4(p_a + p_b - \sum_f p_f)]^2}^{(2\pi)^4 \delta^4(p_a + p_b - \sum_f p_f) VT \text{ by (5.85)}}}{(2p_a^0 V)(2p_b^0 V) \prod_f (2p_f^0 V)} |\mathcal{M}|^2 \prod_f \frac{V}{(2\pi)^3} d^3 p_f \\ &= \frac{(2\pi)^4}{2p_a^0 2p_b^0 \, v} |\mathcal{M}|^2 \delta^4(p_a + p_b - \sum_f p_f) \prod_f \frac{d^3 p_f}{(2\pi)^3 2p_f^0}, \end{aligned} \tag{5.271}$$

or

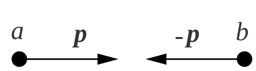
$$\boxed{d\sigma = \frac{(2\pi)^4}{4E_a E_b \, v} |\mathcal{M}|^2 d\Phi_n} \tag{5.272}$$

where the n-body Lorentz-invariant phase space is given by (5.109) with $P = p_a + p_b$ (the total 4-momentum of the system).

The factor $E_a E_b v$ is sometimes called the 'flux factor', and they can be written in two typical cases as

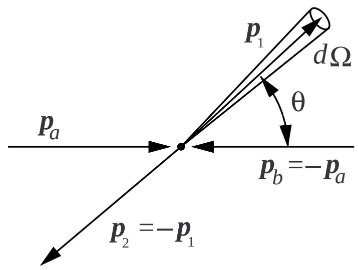


$$(b \text{ at rest}) \quad E_a E_b v = \overbrace{E_b}^{m_b} \overbrace{E_a v}^{|\vec{p}_a|} = m_b |\vec{p}_a| \tag{5.273}$$



$$\begin{aligned} (\text{C.M.}) \quad E_a E_b v &= E_a E_b (|v_a| + |v_b|) = \overbrace{E_a |v_a|}^{|\vec{p}_a|} E_b + E_a \overbrace{E_b |v_b|}^{|\vec{p}_b|} \\ &= \overbrace{(E_a + E_b)}^{M \,:\, \text{invariant mass}} |\vec{p}| = M |\vec{p}|. \end{aligned} \tag{5.274}$$

Following the calculation of the two-body phase space in the C.M. frame (5.96), but this time without integrating over the direction of $\vec{p}_1$, we obtain (left as an exercise)



$$d\Phi_2 = \frac{|\vec{p}_1|}{(2\pi)^6 4M} d\Omega \quad \text{(C.M.)}\,, \tag{5.275}$$

where $M = E_a + E_b$ is the invariant mass or the total energy in the C.M. frame, and $d\Omega = d\phi d\cos\theta$ is the angular element of $\vec{p}_1$ where (θ, ϕ) is the polar coordinates of $\vec{p}_1$ with respect to $\vec{p}_a$. Together with the cross section formula (5.272) and $E_a E_b v = M|\vec{p}_a|$, we have

$$\boxed{\frac{d\sigma}{d\Omega} = \frac{|\mathcal{M}|^2}{(8\pi M)^2} \frac{|\vec{p}_1|}{|\vec{p}_a|} \quad (a + b \to 1 + 2,\ \text{C.M.})\,.} \tag{5.276}$$

If the above cross section does not depend on the azimuthal angle ϕ, then it is a function only of θ only, or

$$t \stackrel{\text{def}}{\equiv} (p_1 - p_a)^2 = m_1^2 + m_a^2 + 2(E_1 E_a - |\vec{p}_1||\vec{p}_a| \cos\theta)\,. \tag{5.277}$$

Changing the variable from θ to t, we obtain (left as an exercise)

$$\boxed{\frac{d\sigma}{dt} = \frac{|\mathcal{M}|^2}{16\pi\lambda(M^2, m_a^2, m_b^2)} \quad (a + b \to 1 + 2)} \tag{5.278}$$

with

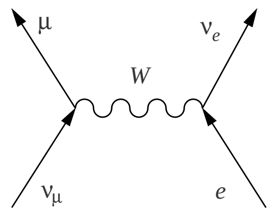
$$\lambda(x, y, z) \stackrel{\text{def}}{\equiv} x^2 + y^2 + z^2 - 2xy - 2yz - 2zx\,, \tag{5.279}$$

and this formula is valid in any frame. For completeness, we give a general 2-body formula for the case b is at rest:

$$\boxed{\frac{d\sigma}{d\Omega_1} = \frac{|\vec{p}_1||\mathcal{M}|^2}{64\pi m_b |\vec{p}_a| \left[m_b + E_a(1 - \frac{\beta_a}{\beta_1}\cos\theta)\right]} \quad (a + b \to 1 + 2,\ b \text{ at rest})\,.} \tag{5.280}$$

Exercise 5.5 *Verify (5.278) and (5.280).*

Let's get back to the scattering $\nu_\mu e^- \to \mu^- \nu_e$. This process occurs through W exchange between $e\nu_e$ current and $\nu_\mu\mu$ current. The responsible interaction is a part of the general W-lepton couplings (5.240):



$$\mathcal{L}_{\rm int} = \frac{g}{\sqrt{2}}\Big(i_\alpha(x) + j_\alpha(x)\Big)W^\alpha(x) + h.c. \quad \text{with} \begin{cases} i_\alpha = \bar\nu_e\gamma_\alpha P_L e \\ j_\alpha = \bar\nu_\mu\gamma_\alpha P_L\,\mu \end{cases}, \tag{5.281}$$

where we have defined i_α to be the $e\nu_e$ current and j_α the $\nu_\mu\mu$ current. The initial and final states are

$$|i\rangle = a_e^\dagger a_{\nu_\mu}^\dagger|0\rangle\,, \quad |f\rangle = a_\mu^\dagger a_{\nu_e}^\dagger|0\rangle\,; \tag{5.282}$$

thus, the non-vanishing term in $S_{fi} = \langle f|S|i\rangle$ should contain $a_\mu^\dagger a_{\nu_e}^\dagger a_{\nu_\mu} a_e$. When the fields are momentum-expanded in $\mathcal{L}_{\rm int}$, however, each term contains a product of three creation or annihilation operators; so the first-order term in the Dyson series vanishes. Let's try the second-order term

$$\begin{aligned} S &= \frac{(-i)^2}{2}\int dt\,dt'\,T(h(t)h(t')) \\ &= -\frac{1}{2}\int dt\,dt'\,T\Big(\int d^3x\mathcal{H}_{\rm int}(t,\vec x)\int d^3x'\mathcal{H}_{\rm int}(t',\vec x')\Big) \\ &\quad \text{(using the linearity of } T\text{-product)} \\ &= -\frac{1}{2}\int d^4x\,d^4x'\,T(\mathcal{H}_{\rm int}(x)\mathcal{H}_{\rm int}(x'))\,. \end{aligned} \tag{5.283}$$

Assuming that there is no time derivatives in $\mathcal{L}_{\rm int}$, we have $\mathcal{L}_{\rm int} = -\mathcal{H}_{\rm int}$; then the S-matrix element is

$$S_{fi} = -\frac{1}{2}\int d^4x\,d^4x'\langle f|T(\mathcal{L}_{\rm int}(x)\mathcal{L}_{\rm int}(x')|i\rangle \quad \text{(second order)}\,. \tag{5.284}$$

Using the interaction Lagrangian (5.281) and writing out the hermitian conjugate parts, we have

$$\begin{aligned} &\langle f|T(\mathcal{L}_{\rm int}\mathcal{L}'_{\rm int})|i\rangle \\ &\quad = \frac{g^2}{2}\langle 0|a_\mu a_{\nu_e} T\Big(\Big[(\boxed{i_\alpha} + j_\alpha)W^\alpha + (i_\alpha^\dagger + \boxed{j_\alpha^\dagger})W^{\alpha\dagger}\Big] \\ &\qquad\qquad \times\Big[(\boxed{i'_\beta} + j'_\beta)W^{\beta\prime} + (i'^\dagger_\beta + \boxed{j'^\dagger_\beta})W^{\beta\prime\dagger}\Big]\Big)a_e^\dagger a_{\nu_\mu}^\dagger|0\rangle\,, \end{aligned} \tag{5.285}$$

where the primed quantities are understood to be functions of x'. There are many terms, but which will survive? Out of the required $a_\mu^\dagger a_{\nu_e}^\dagger a_{\nu_\mu} a_e$, the pair $a_{\nu_e}^\dagger a_e$ is found in $i_\alpha = \bar{\nu}_e \gamma_\alpha P_L e$, and the pair $a^\dagger_\mu a_{\nu_\mu}$ is found in $j^\dagger_\alpha = \bar{\mu}\gamma_\alpha P_L \, \nu_\mu$. Thus, the surviving terms should contain both i_α and $j^\dagger_\alpha$ where they could be functions of x or x' and the indices could be α or β. There are two such terms:

$$\begin{aligned}&\langle f|T(\mathcal{L}_{\rm int}\mathcal{L}'_{\rm int})|i\rangle \\ &= \frac{g^2}{2}\langle 0|a_\mu a_{\nu_e}\Big[\underbrace{T(i_\alpha W^\alpha j'^\dagger_\beta W^{\beta\prime\dagger})}_{(a)} + \underbrace{T(j^\dagger_\alpha W^{\alpha\dagger} i'_\beta W^{\beta\prime})}_{(b)}\Big] a^\dagger_e a^\dagger_{\nu_\mu}|0\rangle \,. \end{aligned} \tag{5.286}$$

Now, the two T-products gives identical results when integrated over x and x', which can be seen as follows: with $f(x) \equiv i_\alpha W^\alpha$ and $g(x) \equiv j^\dagger_\alpha W^{\alpha\dagger}$,

$$(a) = T\Big(f(x)g(x')\Big) \overset{x \leftrightarrow x'}{\longrightarrow} T\Big(f(x')g(x)\Big) = T\Big(g(x)f(x')\Big) = (b)\,, \tag{5.287}$$

where we noted that the ordering of $f(x')$ and $g(x)$ inside the T-product is irrelevant since the ordering is uniquely defined by which of x^0 and $x^{0\prime}$ is larger after all. Thus, (a) and (b) are related by the exchange $x \leftrightarrow x'$, and thus give the same value when integrated over x and x'. We will thus evaluate the first term only and then multiply the result by two which will cancel the factor 1/2 of (5.284).

Remark

The n-th order term of the Dyson series is (by the same derivation that led to the second order expression)

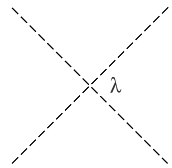
$$S = \frac{(-i)^n}{n\,!}\int d^4x_1 \dots d^4x_n T\Big(\mathcal{H}_{\rm int}(x_1)\dots\mathcal{H}_{\rm int}(x_n)\Big)\,, \tag{5.288}$$

where terms related by permutation of $(x_1, \dots, x_n)$ give identical results as we have seen for the second order case. There are $n\,!$ such terms and it cancels the factor $1/n\,!$ in the expression above. Thus, we can ignore the factor $1/n\,!$ of the Dyson series if we evaluate only one term out of those related by the permutation of $(x_1, \dots, x_n)$. When same field appears m times ($m > 1$) in $\mathcal{L}_{\rm int}$, where ϕ and $\phi^\dagger$ of a charged field are considered different since they contain different creation and annihilation operators, then this protocol (usually) under-counts the number of combinations by factor of $m\,!$ for each space-time integration. For example, if the interaction is $\mathcal{L}_{\rm int} = \lambda\phi^4$, then the first order matrix element for $1+2 \to 3+4$ would be

$$S_{fi} = i\lambda\int d^4x\,\langle 0|a_{\vec{p}_3}a_{\vec{p}_4}\phi^4(x)a^\dagger_{\vec{p}_1}a^\dagger_{\vec{p}_2}|0\rangle \tag{5.289}$$

where there are $m\,!$ ways to match $a_{\vec{p}_3}a_{\vec{p}_4}$ and $a^\dagger_{\vec{p}_1}a^\dagger_{\vec{p}_2}$ with those in ϕ^4 each giving identical result. Such a factor is usually taken into account by re-defining (augumenting) the coupling constant by the factor $m\,!$: e.g.

$$\mathcal{L}_{\rm int} = \frac{1}{4\,!}\lambda\phi^4\,. \tag{5.290}$$



Then, if we take the vertex factor to be simply λ and proceed with the Feynman rules, the factor $4\,!$ above will be properly accounted for. There are exceptions, however, which occur when there are extra symmetries in the diagram. Such symmetry factors will be dealt with when we encounter them. ▮

Note that if one operator, say $A_k(t_k)$, commutes with all others in a time-ordered product $T(A_1(t_1)\ldots A_n(t_n))$, then it can come out of the time-ordered product:

$$\begin{aligned} T\Big(A_1(t_1)\ldots A_n(t_n)\Big) &= A_{i_1}(t_{i_1})\ldots A_{i_n}(t_{i_n}) \qquad (t_{i_1}\geq\ldots\geq t_{i_n}) \\ &= A_k(t_k)\;A_{i_1}(t_{i_1})\ldots \cancel{A_k(t_k)}\ldots A_{i_n}(t_{i_n}) \qquad (t_{i_1}\geq\ldots\geq t_{i_n}) \\ &= A_k(t_k)\;T\Big(A_{i_1}(t_{i_1})\ldots \cancel{A_k(t_k)}\ldots A_{i_n}(t_{i_n})\Big)\,. \end{aligned} \tag{5.291}$$

In $T(i_\alpha W^\alpha j'^\dagger_\beta W^{\beta\prime\dagger})$, i_α and $j'^\dagger_\beta$ commute with each other and with W fields; thus, i_α and $j'^\dagger_\beta$ can come out of the T-product. Then, the first term in (5.286) is

$$\begin{aligned} &\langle 0|a_\mu a_{\nu_e} i_\alpha j'^\dagger_\beta T(W^\alpha W^{\beta\prime\dagger}) a^\dagger_e a^\dagger_{\nu_\mu}|0\rangle \\ &\quad = \langle 0|T(W^\alpha \underbrace{W^{\beta\prime\dagger})\; a_\mu a_{\nu_e}}_{1=\sum_i |i\rangle\langle i|} i_\alpha j'^\dagger_\beta a^\dagger_e a^\dagger_{\nu_\mu}|0\rangle \\ &\quad = \langle 0|T(W^\alpha W^{\beta\prime\dagger})|0\rangle\langle 0|a_\mu a_{\nu_e} i_\alpha j'^\dagger_\beta a^\dagger_e a^\dagger_{\nu_\mu}|0\rangle\,. \end{aligned} \tag{5.292}$$

We have inserted $1=\sum_i |i\rangle\langle i|$, where i runs over all basis states, of which only $|0\rangle\langle 0|$ survives as can be seen as follows: There are only spin-1 fields to its left (call it A) and only fermion fields to its right (call it B). Then, in the matrix element

$$\langle 0|A|i\rangle\langle i|B|0\rangle\,, \tag{5.293}$$

if $|i\rangle$ contains any spin-1 particle, the corresponding annihilation operator in $\langle i|$ faces the vacuum to its right and the matrix element vanishes.Similarly, if $|i\rangle$ contains any fermion, then the corresponding creation operator faces the vacuum to its left and the matrix element vanishes. Thus, the state $|i\rangle$ cannot contain any spin-1 particle nor fermion leaving the vacuum state as the only non-zero possibility. By the same technique, we can write the second factor in (5.292) as a product of two currents:

$$
\begin{aligned}
\langle 0|a_\mu a_{\nu_e} i_\alpha j'^\dagger_\beta a^\dagger_e a^\dagger_{\nu_\mu}|0\rangle &= \langle 0|a_{\nu_e} i_\alpha \underbrace{a^\dagger_e\, a_\mu}_{1=\sum_i |i\rangle\langle i|} j'^\dagger_\beta a^\dagger_{\nu_\mu}|0\rangle \\
&= \langle 0|a_{\nu_e} i_\alpha a^\dagger_e|0\rangle\langle 0|a_\mu j'^\dagger_\beta a^\dagger_{\nu_\mu}|0\rangle \\
&= \langle \nu_e|i_\alpha|e\rangle\langle \mu|j'^\dagger_\beta|\nu_\mu\rangle\,, \qquad (5.294)
\end{aligned}
$$

where we have written

$$
|\nu_\mu\rangle \stackrel{\text{def}}{\equiv} a^\dagger_{\nu_\mu}|0\rangle\,, \quad |\mu\rangle \stackrel{\text{def}}{\equiv} a^\dagger_\mu|0\rangle\,, \quad \text{etc.} \qquad (5.295)
$$

Thus, S_{fi} is now

$$
S_{fi} = -\frac{g^2}{2}\int d^4x\, d^4x' \langle 0|T(W^\alpha W^{\beta\prime\dagger})|0\rangle\, \langle \nu_e|i_\alpha|e\rangle\, \langle \mu|j'^\dagger_\beta|\nu_\mu\rangle\,. \qquad (5.296)
$$

As we will see below, this form of transition amplitude has an intuitive physical interpretation. If we divide the W field into the creation and annihilation parts

$$
W^\alpha = W^\alpha_a + W^{\alpha\dagger}_b \quad \text{with} \quad \begin{cases} W^\alpha_a \equiv \sum_{\vec p\lambda} a_{\vec p\lambda} h^\alpha_{\vec p\lambda} \\ W^\alpha_b \equiv \sum_{\vec p\lambda} b_{\vec p\lambda} h^\alpha_{\vec p\lambda} \end{cases}, \qquad (5.297)
$$

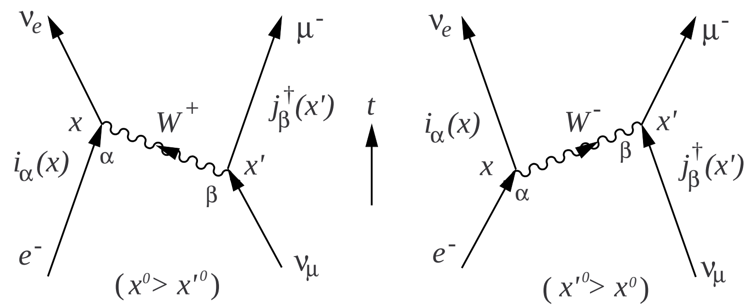
then, using the definition of the time-ordered product,

$$
\begin{aligned}
&\langle 0|T(W^\alpha W^{\beta\prime\dagger})|0\rangle \\
&\quad = \langle 0|W^\alpha W^{\beta\prime\dagger}|0\rangle\theta(x^0 - x^{0\prime}) + \langle 0|W^{\beta\prime\dagger} W^\alpha|0\rangle\theta(x^{0\prime} - x^0) \\
&\quad = \langle 0|W^\alpha_a W^{\beta\prime\dagger}_a|0\rangle\theta(x^0 - x^{0\prime}) + \langle 0|W^{\beta\prime}_b W^{\alpha\dagger}_b|0\rangle\theta(x^{0\prime} - x^0)\,, \qquad (5.298)
\end{aligned}
$$

where in the last line, we have kept only the terms in which creation operators faces the vacuum on their right or annihilation operators faces the vacuum on their left. Recalling that, in the case of scalar field, $\langle 0|\phi_a(x')\phi^\dagger_a(x)|0\rangle$ can be interpreted as the amplitude for the particle to be created at x and found at x', we then naturally interpret $\langle 0|T(W^\alpha W^{\beta\prime\dagger})|0\rangle$ as the amplitude to

$$
\begin{aligned}
&\begin{pmatrix} \text{create } \beta\text{-component of } W^+ \text{ at } x' \text{ and} \\ \text{annihilate } \alpha\text{-component of } W^+ \text{ at } x \end{pmatrix} \quad \text{if } x^0 > x^{0\prime}, \\
&\begin{pmatrix} \text{create } \alpha\text{-component of } W^- \text{ at } x \text{ and} \\ \text{annihilate } \beta\text{-component of } W^- \text{ at } x' \end{pmatrix} \quad \text{if } x^{0\prime} > x^0.
\end{aligned} \qquad (5.299)
$$

Then, the amplitude S_{fi} can be graphically interpreted as below:



(5.300)

One sees that when $x^0 > x^{0\prime}$, W^+ is emitted from the $\nu_\nu \to \mu^-$ current at x' and absorbed by the $e^- \to \nu_e$ current at x. When $x^0 < x^{0\prime}$, W^- is emitted from the $e^- \to \nu_e$ current at x and absorbed by the $\nu_\mu \to \mu^-$ current at x'. Note that at each vertex, the charge is conserved. Also, the α-component of the $e^- \to \nu_e$ current couples to the α-component of W, and the β-component of the $\nu_\mu \to \mu^-$ current couples to the β-component of W. The amplitude is then summed over α and β and integrated over x and x'. The propagation amplitude $\langle 0|T(W^\alpha(x)W^{\beta\dagger}(x'))|0\rangle$ is called the *Feynman propagator* for a spin-1 particle.

Let's evaluate the Feynman propagator $\langle 0|T(W^\alpha(x)W^{\beta\dagger}(x'))|0\rangle$ so that we can actually calculate the transition amplitude S_{fi}. The first term in (5.298) (apart from the θ function) is

$$\begin{aligned}
\langle 0|W^\alpha_a(x)W^{\beta\dagger}_a(x')|0\rangle &= \langle 0|\sum_{\vec p\lambda} a_{\vec p\lambda}h^\alpha_{\vec p\lambda}(x)\sum_{\vec p'\lambda'} a^\dagger_{\vec p'\lambda'}h^{\beta*}_{\vec p'\lambda'}(x')|0\rangle \\
&= \sum_{\substack{\vec p\lambda \\ \vec p'\lambda'}} h^\alpha_{\vec p\lambda}(x)h^{\beta*}_{\vec p'\lambda'}(x')\underbrace{\langle 0|a_{\vec p\lambda}a^\dagger_{\vec p'\lambda'}|0\rangle}_{\delta_{\vec p,\vec p'}\delta_{\lambda,\lambda'}} \\
&= \underbrace{\sum_{\vec p}}_{\frac{V}{(2\pi)^3}\int d^3p} \overbrace{\sum_\lambda \epsilon^\alpha_{\vec p\lambda}\epsilon^{\beta*}_{\vec p\lambda}}^{-g^{\alpha\beta}+\frac{p^\alpha p^\beta}{m^2}} \underbrace{e_{\vec p}(x)e^*_{\vec p}(x')}_{\frac{e^{-ip\cdot(x-x')}}{2p^0V}} \\
&= \frac{1}{(2\pi)^3}\int \frac{d^3p}{2p^0}\Big(-g^{\alpha\beta}+\frac{p^\alpha p^\beta}{m^2}\Big)e^{-ip\cdot(x-x')}\,. \qquad (5.301)
\end{aligned}$$

We obtain the second term by the exchanges $a \leftrightarrow b$, $\alpha \leftrightarrow \beta$, and $x \leftrightarrow x'$, which amounts to the simple sign change of the exponent:

$$\langle 0|W^{\beta\prime}_b W^{\alpha\dagger}_b|0\rangle = \frac{1}{(2\pi)^3}\int \frac{d^3p}{2p^0}\Big(-g^{\alpha\beta}+\frac{p^\alpha p^\beta}{m^2}\Big)e^{ip\cdot(x-x')}\,. \qquad (5.302)$$

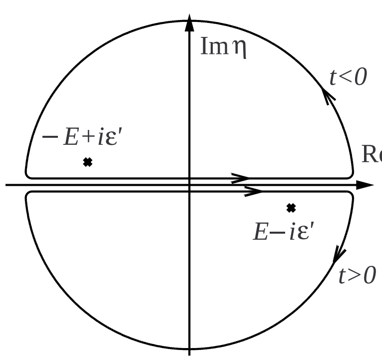
Thus, $\langle 0|T\Big(W_a^\alpha(x)W_a^{\beta\dagger}(x')\Big)|0\rangle$ is a function of $z \equiv x-x'$, which we denote by $iD_F^{\alpha\beta}(z)$. Combining (5.301) and (5.302),

$$\begin{aligned}
iD_F^{\alpha\beta}(z) &\stackrel{\text{def}}{\equiv} \langle 0|T\Big(W^\alpha(x)W^{\beta\dagger}(x')\Big)|0\rangle \qquad (5.303)\\
&= \int \frac{d^3p}{(2\pi)^3 2p^0}\Big[\Big(-g^{\alpha\beta}+\frac{p^\alpha p^\beta}{m^2}\Big)e^{-ip\cdot z}\theta(z^0) + \underbrace{\Big(-g^{\alpha\beta}+\frac{p^\alpha p^\beta}{m^2}\Big)e^{ip\cdot z}}_{\text{relabel } \vec{p}\to-\vec{p}:\ \big(-g^{\alpha\beta}+\frac{p_\alpha p_\beta}{m^2}\big)e^{ip^0z^0+i\vec{p}\cdot\vec{z}}}\theta(-z^0)\Big]\\
&= \int \frac{d^3p}{(2\pi)^3}\frac{e^{i\vec{p}\cdot\vec{z}}}{2p^0}\Big[\Big(-g^{\alpha\beta}+\frac{p^\alpha p^\beta}{m^2}\Big)e^{-ip^0z^0}\theta(z^0) + \Big(-g^{\alpha\beta}+\frac{p_\alpha p_\beta}{m^2}\Big)e^{ip^0z^0}\theta(-z^0)\Big].
\end{aligned}$$

Note that the indeces α and β changed from superscripts to subscripts on $p^\alpha p^\beta$ upon relabeling $\vec{p}\to-\vec{p}$ and that p^0 is constrained to $\sqrt{\vec{p}^{\,2}+m^2}$. The θ functions can be nicely taken care of using complex analysis. To do so, we will first prove the identity

$$I \equiv \int_{-\infty}^{\infty}\frac{d\eta}{2\pi}\frac{f(\eta)e^{-i\eta t}}{\eta^2-E^2+i\epsilon} = \frac{-i}{2E}\Big(f(E)e^{-iEt}\theta(t)+f(-E)e^{iEt}\theta(-t)\Big) \qquad (5.304)$$

where t, E, ϵ are real, $E, \epsilon > 0$, ϵ is small, and η is a complex variable. The integrand has two poles: $(E-i\epsilon')$ and $-(E-i\epsilon')$:



$$\frac{1}{\eta^2-E^2+i\epsilon} = \frac{1}{\eta^2-(E-i\epsilon')^2} = \frac{1}{(\eta-(E-i\epsilon'))(\eta+(E-i\epsilon'))}, \qquad (5.305)$$

where $\epsilon' \sim \epsilon/2E > 0$. What we want is the integral on the real axis from $-\infty$ to ∞, and in order to use the contour integrals, we have to choose the contour such that the integral on the semicircle vanish. We note that

$$e^{-i\eta t} = e^{-i(\mathrm{Re}\,\eta)t}e^{(\mathrm{Im}\,\eta)t} \to 0 \text{ when } \begin{cases} t>0 \text{ and } \mathrm{Im}\,\eta\to-\infty, \text{ or}\\ t<0 \text{ and } \mathrm{Im}\,\eta\to+\infty.\end{cases} \qquad (5.306)$$

Thus, when $t>0$ we have to take the lower loop which picks up the pole at $E-i\epsilon$, and when $t>0$, we have to take the upper loop which picks up the pole at $-E+i\epsilon$. The clockwise (anti-clockwise) contour integral of some function $F(\eta)$ around a first-order pole η_0 is $-(+)2\pi i\,\mathrm{Res}(\eta_0)$ where the residue is given by

$$\mathrm{Res}(\eta_0) = F(\eta)(\eta-\eta_0)\Big|_{\eta=\eta_0} . \tag{5.307}$$

For our case, the function $F(\eta)$ is

$$F(\eta) = \frac{1}{2\pi}\frac{f(\eta)e^{-i\eta t}}{(\eta-(E-i\epsilon'))(\eta+(E-i\epsilon'))} . \tag{5.308}$$

The integral I is then

$$\begin{aligned}
&(t>0)\quad -2\pi i\,\mathrm{Res}(E-i\epsilon') = -2\pi i\,\frac{1}{2\pi}\frac{f(\eta)e^{-i\eta t}}{\eta+(E-i\epsilon')}\Big|_{\eta=E-i\epsilon'} = \frac{-i}{2E}f(E)e^{-iEt} ,\\
&(t<0)\quad 2\pi i\,\mathrm{Res}(-E+i\epsilon') = 2\pi i\,\frac{1}{2\pi}\frac{f(\eta)e^{-i\eta t}}{\eta-(E-i\epsilon')}\Big|_{\eta=-E+i\epsilon'} = \frac{-i}{2E}f(-E)e^{iEt} ,
\end{aligned} \tag{5.309}$$

which establishes (5.304).

Now, we define

$$f(p^0) \stackrel{\text{def}}{\equiv} -g^{\alpha\beta} + \frac{p^\alpha p^\beta}{m^2} , \tag{5.310}$$

which is a function of p^0 when $\alpha = 0$ or $\beta = 0$, where p^i ($i = 1, 2, 3$) are considered to be constant. Then, the following holds for all values of α and β:

$$f(-p^0) = -g^{\alpha\beta} + \frac{p_\alpha p_\beta}{m^2} , \tag{5.311}$$

where we have lowered the indexes α and β in the second term, which can be verified explicitly for each value of α and β. Using this definition of $f(p^0)$ with $E = p^0$ and $t = z^0$, the identity (5.304) can then be applied to the Feynman propagator (5.303):

$$\begin{aligned}
iD_F^{\alpha\beta}(z) &= \int \frac{d^3p}{(2\pi)^3}\frac{e^{i\vec{p}\cdot\vec{z}}}{2p^0}\Big(f(p^0)e^{-ip^0z^0}\theta(z^0) + f(-p^0)e^{ip^0z^0}\theta(-z^0)\Big)\\
&= \int \frac{d^3p}{(2\pi)^3}e^{i\vec{p}\cdot\vec{z}}\, i\int_{-\infty}^{\infty}\frac{d\eta}{2\pi}\frac{f(\eta)e^{-i\eta z^0}}{\eta^2 - \underbrace{p^{0\,2}}_{\vec{p}^{\,2}+m^2} + i\epsilon}\\
&= i\int \frac{d^3p\,d\eta}{(2\pi)^4}\frac{f(\eta)e^{-i\eta z^0}e^{i\vec{p}\cdot\vec{z}}}{\eta^2 - \vec{p}^{\,2} - m^2 + i\epsilon} .
\end{aligned} \tag{5.312}$$

At this point, there is no p^0 in sight, not even in $f(\eta)$ where p^0 has been replaced by η in the definition (5.310). Since η is a dummy integration variable, we can call it any way we wish. We choose to call it p^0, which is *no longer constrained to* $p^0 = \sqrt{\vec{p}^2 + m^2}$. Then,

$$
\begin{aligned}
iD_F^{\alpha\beta}(z) &= i\int \frac{d^3p\,dp^0}{(2\pi)^4} \frac{f(p^0)e^{-ip^0z^0}e^{i\vec{p}\cdot\vec{z}}}{p^{0\,2} - \vec{p}^2 - m^2 + i\epsilon} \\
&= i\int \frac{d^4p}{(2\pi)^4} \frac{-g^{\alpha\beta} + \frac{p^\alpha p^\beta}{m^2}}{p^2 - m^2 + i\epsilon}\, e^{-ip\cdot z}\,,
\end{aligned} \tag{5.313}
$$

where we have used the formal definitions

$$
d^4p \stackrel{\text{def}}{\equiv} d^3p\,dp^0 \qquad p^2 \stackrel{\text{def}}{\equiv} p^{0\,2} - \vec{p}^2\,. \tag{5.314}
$$

Thus, the spin-1 Feynman propagator is now

$$
\boxed{
\begin{gathered}
iD_F^{\alpha\beta}(x-x') \stackrel{\text{def}}{\equiv} \langle 0|T\Big(W^\alpha(x)W^{\beta\dagger}(x')\Big)|0\rangle = \int \frac{d^4p}{(2\pi)^4} iD_F^{\alpha\beta}(p)e^{-ip\cdot(x-x')} \\
\text{with} \quad iD_F^{\alpha\beta}(p) = i\frac{-g^{\alpha\beta} + \frac{p^\alpha p^\beta}{m^2}}{p^2 - m^2 + i\epsilon}\,.
\end{gathered}
} \tag{5.315}
$$

We emphasize again that p^0 is an integration variable and not constrained to $p^0 = \sqrt{\vec{p}^2 + m^2}$.

We can now use the spin-1 Feynman propagator to calculate S_{fi} (5.296) for the scattering $\nu_\mu e^- \to \mu^- \nu_e$ and extract the Lorentz-invariant matrix element $\mathcal{M}$. Roughly speaking, what happens is as follows: after collecting exponentials from the fermion normal-mode functions and from the Feynman propagator, the integration over x gives a delta function for the 4-momentum conservation at the W-$e\nu_e$ vertex, and the integration over x' results in another delta function for the 4-momentum conservation at the W-$\nu_\mu\mu$ vertex. Upon performing the integration over the 4-momentum of the W [which is in $iD_F(x-x')$], the two delta functions becomes one that corresponds to the 4-momentum conservation of initial and final states. Then, the propagator in momentum space $D_F^{\alpha\beta}(p)$ will survive into $\mathcal{M}$ together with the u, v spinors for the external fermion legs.

Using the explicit expressions for the currents i_α and j_β (5.281), we have

$$
\begin{aligned}
\langle \nu_e|i_\alpha|e\rangle &= \langle 0|a_{\nu_e}(\bar{\nu}_e\gamma_\alpha P_L e)a_e^\dagger|0\rangle &= \bar{f}_{\nu_e}\gamma_\alpha P_L f_e \\
\langle \mu|j_\beta'^\dagger|\nu_\mu\rangle &= \langle 0|a_\mu(\bar{\mu}'\gamma_\beta P_L \nu_\mu')a_{\nu_\mu}^\dagger|0\rangle &= \bar{f}_\mu'\gamma_\beta P_L f_{\nu_\mu}'
\end{aligned}\,. \tag{5.316}
$$

Then, assigning the 4-momenta as in Figure 5.4, S_{fi} (5.296) is now written as

$$
S_{fi} = -\frac{g^2}{2}\int d^4x\,d^4x' \overbrace{\int \frac{d^4p}{(2\pi)^4} iD_F^{\alpha\beta}(p)e^{-ip\cdot(x-x')}}^{\langle 0|T(W^\alpha W^{\beta\prime\dagger})|0\rangle} \overbrace{\frac{e^{ik'\cdot x}}{\sqrt{2k'^0V}}\frac{e^{-iq\cdot x}}{\sqrt{2q^0V}}\frac{e^{iq'\cdot x'}}{\sqrt{2q'^0V}}\frac{e^{-ik\cdot x'}}{\sqrt{2k^0V}}}^{(\bar{f}_{\nu_e}\gamma_\alpha P_L f_e)(\bar{f}_\mu'\gamma_\beta P_L f_{\nu_\mu}')}
$$

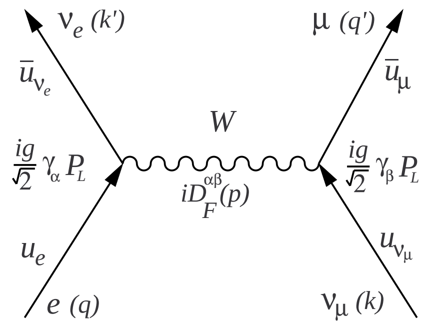


Figure 5.4: The Feynman diagram for the scattering $\nu_\mu e^- \to \mu^- \nu_e$.

$$
\begin{aligned}
& \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \times(\bar{u}_{\nu_e}\gamma_\alpha P_L u_e)(\bar{u}_\mu \gamma_\beta P_L u_{\nu_\mu}) \\
= & \int \frac{d^4p}{(2\pi)^4} \frac{\int d^4x\, e^{-i(p+q-k')\cdot x} \int d^4x' e^{i(p+q'-k)\cdot x'}}{\sqrt{2k'^0 V 2q^0 V 2q'^0 V 2k^0 V}} \\
& \qquad\qquad\qquad\qquad \times \underbrace{\frac{-g^2}{2}(\bar{u}_{\nu_e}\gamma_\alpha P_L u_e) i D_F^{\alpha\beta}(p)(\bar{u}_\mu \gamma_\beta P_L u_{\nu_\mu})}_{\equiv \mathcal{M}} \\
= & \int \frac{d^4p}{(2\pi)^4} \underbrace{(2\pi)^4\delta^4(p+q-k')}_{x \text{ vertex}} \underbrace{(2\pi)^4\delta^4(p+q'-k)}_{x' \text{ vertex}} \frac{\mathcal{M}}{\sqrt{2k'^0 V 2q^0 V 2q'^0 V 2k^0 V}} \\
= & \frac{(2\pi)^4\delta^4(k+q-k'-q')}{\sqrt{2k'^0 V 2q^0 V 2q'^0 V 2k^0 V}} \mathcal{M}\,,
\end{aligned}
\tag{5.317}
$$

which has the standard form of the definition of $\mathcal{M}$ (5.270) and allows us to use the cross section formulas we have derived. The Lorentz-invariant matrix element $\mathcal{M}$

$$
\mathcal{M} \stackrel{\text{def}}{\equiv} \Big(\frac{ig}{\sqrt{2}}\Big)^2 (\bar{u}_{\nu_e}\gamma_\alpha P_L u_e) i \frac{-g^{\alpha\beta} + \frac{p^\alpha p^\beta}{m^2}}{p^2 - m^2 + i\epsilon} (\bar{u}_\mu \gamma_\beta P_L u_{\nu_\mu}) \tag{5.318}
$$

can be obtained directly from the Feynman diagram shown in Figure 5.4 by following the same rules for the external fermions and vertexes as before and assigning the factor $iD_F^{\alpha\beta}(p)$ for the W propagator. Note that the value of p in the propagator is constrained to $p = k' - q = k - q'$ by the delta functions.

Let's evaluate the spin-averaged $|\mathcal{M}|^2$. We will assume that the initial-state electron is unpolarized, but the muon neutrino is assumed to be left-handed since the $V - A$ current is the only known source of neutrinos; namely, when we have a beam of neutrinos, it is a good assumption that they are left-handed. On the other hand, the matrix element above vanishes for a right-handed ν_μ due to the factor $P_L u_{\nu_\mu}$. Consequently, if we sum over the spin of ν_μ, it will properly evaluate the cross section for left-handed ν_μ. Thus, we will sum over all spins and divide by two to account for the unpolarized initial-state electron. Also, we will ignore the masses of the fermions assuming a high-energy scattering. Then, the $p^\alpha p^\beta$ term of the W-propagator vanishes by the same reason as does the $p^\alpha p^\beta$ term of the W spin sum in the $W^+ \to e^+\nu_e$ decay (5.257). The properly spin-averaged matrix element squared is then

$$\overline{|\mathcal{M}|^2} = \frac{1}{2}\Big(\frac{g^2}{2}\Big)^2 \frac{1}{|p^2 - m^2 + i\epsilon|^2} \underbrace{\sum_{\text{spins}} |(\bar{u}_{\nu_e}\gamma_\alpha P_L u_e)(\bar{u}_\mu \gamma^\alpha P_L u_{\nu_\mu})|^2}_{*} \tag{5.319}$$

We have already performed exactly the same spin sum (*) for the $\mu^- \to \nu_\mu e^- \nu_e$ decay. The only difference is that we had a v spinor for ν_e before instead of a u spinor in this case; when summed over spin, however, there is no difference between u and v spinors as long as the particle is massless:

$$\sum_{\vec{s}} v_{\vec{p},\vec{s}}\bar{v}_{\vec{p},\vec{s}} = \not{p} - \cancel{m} = \not{p} + \cancel{m} = \sum_{\vec{s}} u_{\vec{p},\vec{s}}\bar{u}_{\vec{p},\vec{s}} \qquad (\text{massless})\,. \tag{5.320}$$

Comparing with the complex conjugate of (5.125), we see that all that is needed is to substitute

$$p \to q'(\mu)\,, \quad q \to q(e)\,, \quad p' \to k(\nu_\mu)\,, \quad q' \to k'(\nu_e)\,, \tag{5.321}$$

in (5.144) and adjust for the spin average factor of 1/2 used there and the difference between P_L and $(1-\gamma_5)$. Thus, we obtain

$$(*) = 16(q'\cdot k')(k\cdot q) \tag{5.322}$$

The $i\epsilon$ term in the propagator has no effect in this case, and $\overline{|\mathcal{M}|^2}$ is now

$$\overline{|\mathcal{M}|^2} = \frac{1}{2}\Big(\frac{g^2}{2}\Big)^2 \frac{1}{(p^2-m^2)^2} 16(q'\cdot k')(k\cdot q) = 2g^4 \frac{(q'\cdot k')(k\cdot q)}{(p^2-m^2)^2}\,. \tag{5.323}$$

Let's define the Lorentz-invariant parameters s and t as

$$\begin{aligned} s &\equiv (k+q)^2 = (k'+q')^2 = 2k\cdot q = 2k'\cdot q' \\ t &\equiv (k'-q)^2 = (k-q')^2 = p^2 \end{aligned} \tag{5.324}$$

where we have used the 4-momentum conservation $k+q = k'+q'$ and $k^2 = q^2 = k'^2 = q'^2$. Note that s is the invariant mass of the system squared that we called M^2 when we derived the cross section formulas, and t is the same 4-momentum transfer squared that we defined in (5.277). Using the formula $d\sigma/dt = \overline{|\mathcal{M}|^2}/16\pi\lambda(M^2, m_a^2, m_b^2)$ with $\lambda(M^2, m_a^2, m_b^2) = s^2$, we have

$$\frac{d\sigma}{dt} = \frac{1}{16\pi s^2} 2g^4 \frac{(s/2)^2}{(t-m^2)^2} = \frac{g^4}{32\pi(t-m^2)^2} \sim \frac{g^4}{32\pi m^4}\,, \tag{5.325}$$

where the last approximation used $t \ll m^2$ which is a good approximation for most neutrino beams generated by high-energy accelerators. The distribution is essentially flat in t. The range of t can be easily obtained in the C.M. system where all particles in the initial and final states have the same energy $E = M/2$. If the angle between the incoming ν_μ and the outgoing μ^- is θ, then together with $s = 4E^2$,

$$\begin{aligned} t = (k-q')^2 = -2k\cdot q' = -2(E^2 - E^2\cos\theta) = -\frac{s}{2}(1-\cos\theta) \\ \rightarrow \quad 0 > t > -s\,; \end{aligned} \tag{5.326}$$

namely, t distributes uniformly from 0 to $-s$. Thus, the total cross section is (recalling that m is the W mass)

$$\sigma(\nu_\mu e^- \to \mu^- \nu_e) = \frac{g^4}{32\pi m_W^4} s \qquad (t \ll m_W^2, \quad \text{massless fermions})\,. \tag{5.327}$$

Let's work out the number of interaction for a hypothetical case of 10^{13} ν_μ's at 200 GeV hitting an iron target of thickness 10 m. With $g = 0.65$, $m_W = 80$ GeV, and $m_e = 5.11 \times 10^{-4}$ GeV, we have

$$\begin{aligned} s = 2k\cdot q = 2E_{\nu_e} m_e = 0.204\ \text{GeV}^2 \\ \rightarrow \quad \sigma(\nu_\mu e^- \to \mu^-\nu_e) = 8.8 \times 10^{-12}\ \text{GeV}^{-2}\,. \end{aligned} \tag{5.328}$$

The expected number of interaction per projectile is

$$N = n(\text{cm}^{-3})\,\sigma(\text{cm}^2)L(\text{cm})\,, \tag{5.329}$$

where n and L are the density and the length of the target. Thus, we have to convert the unit of cross section from GeV^{-2} to cm^2. Since σ was evaluated in the unit system where $\hbar = c = 1$, and $\hbar c$ has the value

$$\hbar c = 1.9732 \times 10^{-14}\ \text{GeV}\cdot\text{cm}\,, \tag{5.330}$$

if we pick the unit of energy to be GeV, then the unit of length (GeV^{-1}) should correspond to 1.9732×10^{-14} cm in order to make $\hbar c$ unity. Thus, the unit conversion is

$$\sigma(\text{cm}^2) = \sigma(\text{GeV}^{-2}) \times (1.9732 \times 10^{-14})^2 = 3.4 \times 10^{-39}\text{cm}^2\,. \tag{5.331}$$

The electron density of the iron is $n = 2.2 \times 10^{24}/\text{cm}^3$; then, the total number of interactions is

$$\begin{aligned} (\#\nu_\mu)N &= (\#\nu_\mu)n\sigma L \\ &= 10^{13}\cdot 2.2\times 10^{24}(\text{cm}^{-3})\cdot 3.4\times 10^{-39}(\text{cm}^2)\cdot 10^3(\text{cm}) \sim 75\,, \end{aligned} \tag{5.332}$$

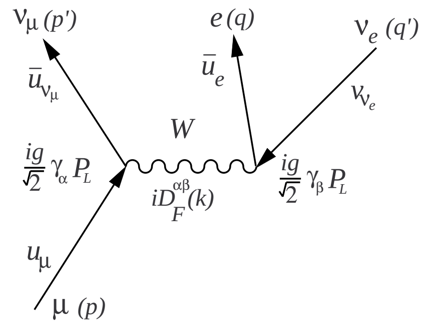


Figure 5.5: The Feynman diagram for the muon decay $\mu^- \to \nu_\mu e^- \bar{\nu}_e$ occurring through a W exchange.

which is not a bad number to start planning the experiment!

The muon decay revisited - extraction of g

The muon decay $\mu^- \to \nu_\mu e^- \nu_e$ that we have studied earlier using the effective interaction (5.121) occurs through an exchange of W as shown in Figure 5.5, from which the Lorentz-invariant matrix element can be read off:

$$\mathcal{M} \stackrel{\text{def}}{\equiv} \Big(\frac{ig}{\sqrt{2}}\Big)^2 (\bar{u}_{\nu_\mu}\gamma_\alpha P_L u_\mu) i \frac{-g^{\alpha\beta} + \frac{k^\alpha k^\beta}{m_W^2}}{k^2 - m_W^2 + i\epsilon} (\bar{u}_e \gamma_\beta P_L v_{\nu_e}) \tag{5.333}$$

Since $\mathcal{M}$ is Lorentz-invariant, let's roughly evaluate the sizes of parameters in the muon rest frame. The 4-momentum $k = q + q'$ is the 4-momentum of the $e^-\bar{\nu}_e$ system, and the maximum value of $|k^\alpha|$ occurs when the momentum of ν_μ is zero and all energy of μ^- has to be carried by the $e^-\bar{\nu}_e$ system; namely, $k^0 = m_\mu$ in this configuration and it is the maximum size that any component of k can take. Thus,

$$\left|\frac{k^\alpha k^\beta}{m_W^2}\right|, \left|\frac{k^2}{m_W^2}\right| \le \Big(\frac{m_\mu}{m_W}\Big)^2 \sim 10^{-5}\,. \tag{5.334}$$

Thus, to a good accuracy,

$$\frac{-g^{\alpha\beta} + \frac{k^\alpha k^\beta}{m_W^2}}{k^2 - m_W^2 + i\epsilon} \approx \frac{g^{\alpha\beta}}{m_W^2}\,; \tag{5.335}$$

namely,

$$\mathcal{M} = \frac{ig}{8m_W^2}(\bar{u}_{\nu_\mu}\gamma_\alpha(1-\gamma_5)u_\mu)(\bar{u}_e\gamma^\alpha(1-\gamma_5)v_{\nu_e}) \tag{5.336}$$

this is exactly the same matrix element obtained from the effective interaction with the identification

$$\boxed{\frac{g^2}{8m_W^2} = \frac{G_F}{\sqrt{2}}}\,. \tag{5.337}$$

Since we know the value of the Fermi coupling constant G_F from the muon life time, we can extract the value of the universal coupling constant g:

$$G_F = 1.1664 \times 10^{-5}(\mathrm{GeV}^{-2}) \quad \rightarrow \quad g = 0.65 \tag{5.338}$$

which is the value we have been using.

Propagators for spin-1/2 and spin-0 particles

Just as in the case of spin-1 particles, when a spin-1/2 or spin-0 particle is created and then absorbed in the course of a process - namely, when it does not appear as an external leg - it has an effect of propagating a particle from some space-time point x' to another space-time point x when $x^0 > x^{0\prime}$ and propagating an antiparticle in the opposite direction if $x^{0\prime} > x^0$. They are again expressed as vacuum expectation values of time-ordered products of fields as in the spin-1 case and called the Feynman propagators for spin-1/2 and spin-0 particles, respectively. Following the same procedure as in the spin-1 case, the spin-1/2 Feynman propagator is

$$\boxed{\begin{array}{c} iS_{Fnm}(x-x') \stackrel{\text{def}}{\equiv} \langle 0|T\Big(\psi_n(x)\bar{\psi}_m(x')\Big)|0\rangle = \displaystyle\int \frac{d^4p}{(2\pi)^4} iS_{Fnm}(p)e^{-ip\cdot(x-x')} \\ \text{with} \quad iS_{Fnm}(p) = i\dfrac{(\not{p}+m)_{nm}}{p^2-m^2+i\epsilon}\,, \end{array}} \tag{5.339}$$

where p is the 4-momentum that would be carried by the fermion propagating forward in time (not the anti-fermion), and the spin-0 Feynman propagator is

$$\boxed{\begin{array}{c} i\Delta_F(x-x') \stackrel{\text{def}}{\equiv} \langle 0|T\Big(\phi(x)\phi^\dagger(x')\Big)|0\rangle = \displaystyle\int \frac{d^4p}{(2\pi)^4} i\Delta_F(p)e^{-ip\cdot(x-x')} \\ \text{with} \quad i\Delta_F(p) = \dfrac{i}{p^2-m^2+i\epsilon}\,, \end{array}} \tag{5.340}$$

which is valid for both hermitian and non-hermitian (charged) fields.

Exercise 5.6 *Feynman propagators.*
(a) For a hermitian spin-0 field $\phi(x)$, show that the vacuum expectation value of the time-ordered product of $\phi(x)$ and $\phi(y)$ (the Feynman propagator for spin-0 particle) can be written as

$$\begin{aligned} i\Delta_F(x-y) &\equiv \langle 0|T(\phi(x)\phi(y))|0\rangle \\ &= \int \frac{d^3p}{(2\pi)^3 2p^0}\Big[e^{-ip\cdot(x-y)}\theta(x^0-y^0) + e^{ip\cdot(x-y)}\theta(y^0-x^0)\Big] \end{aligned} \tag{5.341}$$

where $p^0 \equiv \sqrt{\vec{p}^2 + m^2}$. Then use the identity

$$\frac{1}{2E}\Big(f(E)e^{-iEt}\theta(t) + f(-E)e^{iEt}\theta(-t)\Big) = \int_{-\infty}^{\infty} \frac{d\eta}{2\pi} \frac{i\, f(\eta)e^{-i\eta t}}{\eta^2 - E^2 + i\epsilon} \tag{5.342}$$

to write it in the form given in (5.340) with $\phi^\dagger = \phi$. The Feynman propagator for a hermitian spin-0 field is the same as that for a charged spin-0 field.
(b) Propagator for Dirac field: Verify

$$\begin{aligned} \langle 0|\psi_j(x)\bar{\psi}_k(y)|0\rangle &= \int \frac{d^3p}{(2\pi)^3 2p^0} (\not{p} + m)_{jk} e^{-ip\cdot(x-y)} \\ \langle 0|\bar{\psi}_k(y)\psi_j(x)|0\rangle &= \int \frac{d^3p}{(2\pi)^3 2p^0} (\not{p} - m)_{jk} e^{ip\cdot(x-y)} \end{aligned} \tag{5.343}$$

Then, show that the Feynman propagator for a spin-1/2 particle is given by

$$\begin{aligned} iS_{F\,jk}(x-y) &\equiv \langle 0|T(\psi_j(x)\bar{\psi}_k(y))|0\rangle \\ &= \langle 0|\psi_j(x)\bar{\psi}_k(y)|0\rangle\theta(x^0 - y^0) - \langle 0|\bar{\psi}_k(y)\psi_j(x)|0\rangle\theta(y^0 - x^0) \\ &= \int \frac{d^4p}{(2\pi)^4} \frac{i(\not{p} + m)_{jk}}{p^2 - m^2 + i\epsilon} e^{-ip\cdot(x-y)}\,. \end{aligned} \tag{5.344}$$

Note the minus sign for the T-product for femion fields.
(c) Verify

$$(\not{p} - m)^{-1} = \frac{\not{p} + m}{p^2 - m^2}\,; \tag{5.345}$$

thus, the spin-1/2 propagator in momentum space is often written as

$$iS_F(p) = \frac{i}{\not{p} - m + i\epsilon} \stackrel{\text{def}}{\equiv} i\frac{\not{p} + m}{p^2 - m^2 + i\epsilon}\,. \tag{5.346}$$

The exponential factor $e^{-ip\cdot(x-x')}$ is common to the propagators for spin-0, spin-1/2, and spin-1 particle, and when integrated over x and x', it becomes part of the delta functions that represent 4-momentum conservation at ‘each end’ of the propagator. Thus, the rule is that when we have an internal line in Feynman diagram (i.e. propagator), we assign the following factor:

$$\begin{aligned} &\text{spin-0:} && i\Delta_F(p) = \frac{i}{p^2 - m^2 + i\epsilon}\,, \\ &\text{spin-1/2:} && iS_F(p) = i\frac{\not{p} + m}{p^2 - m^2 + i\epsilon}\,, \\ &\text{spin-1:} && iD_F^{\alpha\beta}(p) = i\frac{-g^{\alpha\beta} + \frac{p^\alpha p^\beta}{m^2}}{p^2 - m^2 + i\epsilon}\,. \end{aligned} \tag{5.347}$$

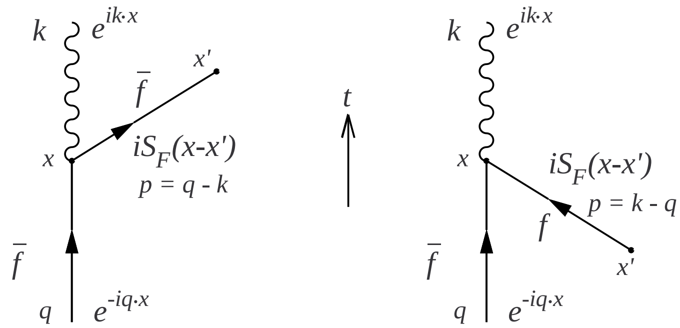


Figure 5.6: The sign of the 4-momentum in a fermion propagator is defined to be the one carried by fermion propagating forward in time, not by anti-fermion.

Since the 4-momentum is to be conserved at each end of the propagator, what value to take for the 4-momentum p in the propagators in momentum space is usually quite obvious - except for the sign which depends on whether the propagator is thought of as a particle propagating *forward* in time or as an anti-particle propagating again *forward* in time. The sign of p does not matter for the spin-0 and spin-1 propagators which are symmetric under $p \to -p$ as can be seen in (5.347). On the other hand, the sign does matter for the spin-1/2 propagator since the relative sign of p and m in the numerator $\not{p} + m$ makes a difference.

As an example, consider a case where initial state anti-fermion emits a vector meson at x and connects there to a fermion propagator (Figure 5.6). The other end of the propagator (x') would connect to a fermion line and a vector, etc., but it is irrelevant for us now. The propagator $iS_F(x - x')$ as defined in (5.339) propagates an anti-fermion from x to x' when $x'^0 > x^0$ or propagates a fermion from x' to x when $x'^0 < x^0$ which is exactly what is needed in this case. If the incoming particle were a fermion instead of anti-fermion, we would have used $iS_F(x' - x)$ which would propagate a fermion from x to x' when $x'^0 > x^0$. What p to use in $iS_F(p)$ is determined by the delta function that arises when integrated over x. The incoming antifermion should be matched with $b_{\vec{q},\vec{s}}\,\bar{g}_{\vec{q},\vec{s}}(x)$, thus it comes with $e^{-iq\cdot x}$. Similarly, the outgoing vector comes with $e^{ik\cdot x}$, and $iS_F(x - x')$ contains $e^{-ip\cdot x}$. Thus we obtain $\delta^4(q - k + p)$ upon integration over x; namely, we should use $p = k - q$ which is the 4-momentum of the *fermion* propagating forward in time (the diagram on the right of Figure 5.6). If the incoming particle were a fermion, then the only modification in the exponential factors above comes from $iS_F(x' - x)$ which now has $e^{ip\cdot x}$ instead of $e^{-ip\cdot x}$. Thus, the 4-momentum to use in $iS_F(p)$ would be $p = q - k$ which again is the 4-momentum of the fermion propagating forward in time (the diagram on the left of Figure 5.6 where $\bar{f}$ is changed to f). The rule is then the 4-momentum of a fermion propagator should be the one carried by a fermion propagating forward in time.

In the massless limit, the spin-0 and spin-1/2 propagators in (5.347) do not encounter diffculties; the spin-1 propagator, however, requires some care since the term $p^\alpha p^\beta/m^2$ in the numerator diverges. As we will see later, only for certain type of theories (where the spin-1 particle couples to a conserved current) the term $p^\alpha p^\beta/m^2$ is strictly zero and thus the spin-1 propagator is well-defined in the massless limit. We will come back to this problem later.

One important feature to note is that even though the time-ordered product is defined in a given Lorentz frame, the expressions above for the Feynman propagators appear to be Lorentz-invariant. To see this more clearly, let's write down the Feynman propagator for the real spin-0 particle as

$$\begin{aligned} \langle 0|T\Big(\phi(x)\phi(x')\Big)|0\rangle &= \langle 0|\phi(x)\phi(x')|0\rangle\theta(x^0 - x^{0\prime}) + \langle 0|\phi(x')\phi(x)|0\rangle\theta(x^{0\prime} - x^0) \\ &= \Delta_+(x - x')\theta(x^0 - x^{0\prime}) + \Delta_+(x' - x)\theta(x^{0\prime} - x^0) \\ &= \Delta_+(x - x')\theta(x^0 - x^{0\prime}) + \Delta_+^*(x - x')\theta(x^{0\prime} - x^0)\,, \qquad (5.348) \end{aligned}$$

where we have used (4.268) and $\Delta_+(-z) = \Delta_+^*(z)$. Namely, the Feynman propagator is obtained from $\Delta_+(z)$ by taking the complex conjuagate when $z^0 < 0$. If for example we boost in the x-direction by a boost factor γ, then the function $\Delta_+(z)$ will still be identical since it depends only on z^2, and the line (actually a plane) $z^0 = 0$ in the original frame corresponds to

$$z^0 = \gamma z^{0\prime} - \eta z^{1\prime} = 0\,, \qquad (5.349)$$

which is a straight line going through the origin whose slope angle never exceeds $\pi/4$. Thus, the time-ordering in the original frame amounts to taking the complex conjugate of $\Delta_+(z)$ below the tilted line in the boosted frame. Since $\Delta_+(z)$ in the space-like region is real (see Figure 4.2), the result does not depend on the slope. Thus, the Feynman propagator is independent of the frame in which the time-ordering is taken. This may be surprising since a propagation of a particle from 0 to z in original frame becomes a propagation of antiparticle from z to 0 in another frame when the sign of z^0 changes under the Lorentz transformation. One sees that the reason why this does not make any difference is because $\Delta_+(z)$ is real in the time-like region.

Before ending this chapter, let's consider what is the meaning of the sign of $i\epsilon$. If one sets the small imaginary part to be $-i\epsilon$ in (5.304) and follow the derivation of the identity, one sees that it only changes the sign of t in the two theta functions, which traces back to the theta functions in the time-ordered product (5.298). Thus, it corresponds to propagating a particle from x' to x if x' is later in time than x and propagating an antiparticle from x to x' if x is later in time than x'; namely, if the sign of $i\epsilon$ is flipped, it will propagate particle or antiparticle backward in time.